

Finite groups with the pp-embedding property

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ABSTRACT – A subset X of a finite group G is called g -independent if there is no proper subset Y of X such that $\langle Y, \Phi(G) \rangle = \langle X, \Phi(G) \rangle$. The group G has the embedding property if every g -independent subset of G can be embedded in a minimal generating set of G . If X is a set of prime power order elements, then we say that G has the pp-embedding property. In this note we classify all finite solvable groups with the pp-embedding property. Moreover we prove that this class is equal to the class of finite solvable groups with the embedding property.

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1. Introduction

All groups considered in this paper are finite. We use standard notations following [4]. In particular if G is a group, then $\Phi(G)$ denotes the Frattini subgroup of G and $F(G)$ denotes the Fitting subgroup of G .

Let G be a group and X be a set of elements of G . We say that X is a g -independent set of G if there is no proper subset Y of X such that $\langle Y, \Phi(G) \rangle = \langle X, \Phi(G) \rangle$. Moreover we say that X is a g -base of G if X is a g -independent generating set of G .

The Burnside Basis Theorem states that in arbitrary finite p -group P every g -independent set of P may be embedded in a g -base of P and all g -bases of P have the same cardinality. It turns out that not only p -groups have such property.

We say that a finite group G :

- has *property B* if its g -bases have the same cardinality ([10]);
- has the *embedding property* if every g -independent set of G can be embedded in a g -base of G ([1, 13]);

- is a *matroid* group if G has property \mathcal{B} and the embedding property.

Groups with the properties presented above have been studied by many authors. A classification of all finite matroid groups was presented in [12]. First results concerning groups with property \mathcal{B} appeared in [10]. In [3] the complete characterization of all such groups was provided. Some of the recent results concerning sets of generators of finite groups appeared in [2].

Groups with the embedding property were first considered in [13]. The following theorem summarizes properties of groups with the embedding property which were proved in [13].

THEOREM 1.1. *Let G be a solvable group and $\Phi(G) = 1$. If G has the embedding property, then $G = F(G) \rtimes H$, where H is an abelian subgroup of G and every subgroup of $F(G)$ is normal in G . In particular G is supersolvable.*

The class of groups with property \mathcal{B} and matroid groups are rather narrow. For example an abelian group is contained in these classes if and only if it is a p -group. Therefore in [7] the definition of property \mathcal{B} was weakened by considering only generating sets in which every element has prime power order. To continue this approach we say that an element $g \in G$ is called a *pp-element* if it is of prime power order. Further we say that a subset $X \subseteq G$ is *pp-independent* if X is a g -independent set of pp-elements and X is a *pp-base* of G if X is a pp-independent generating set of G .

Analogously to above properties, we say that a finite group G

- has *property \mathcal{B}_{pp}* if its pp-bases have the same cardinality;
- has the *pp-embedding property* if every pp-independent set of G can be embedded in a pp-base of G ;
- is a *pp-matroid* group if G has property \mathcal{B}_{pp} and the pp-embedding property.

The class of pp-matroid groups and groups with the pp-basis property, i.e. groups whose all subgroups have property \mathcal{B} , have been completely described (see [9]).

Our purpose is to describe all solvable groups with the pp-embedding property and all solvable groups with the embedding property. Our main result is the following

THEOREM 1.2. *Let G be a solvable group and $\Phi(G) = 1$. Then G has the pp-embedding property if and only if $G = G_1 \times \dots \times G_n$, where every G_i is either*

- a Sylow subgroup of $F(G)$, or*
- $G_i = F(G_i) \rtimes \langle x \rangle$, where x has prime order, every subgroup of $F(G_i)$ is normal in G_i and $C_{F(G_i)}(x) = 1$. Moreover $(|F(G_i)|, |G_j|) = 1$ for $j \neq i$.*

From definitions, we know that every group with property \mathcal{B} has property \mathcal{B}_{pp} , but not conversely. For example from [7, 8] we know that all nilpotent groups have property \mathcal{B}_{pp} (are pp-matroid). On the other hand from [10] we know that

among nilpotent groups only p -groups have property \mathcal{B} (are matroid). So the class of groups with property \mathcal{B}_{pp} (pp-matroid groups) is essentially larger than the class of groups with property \mathcal{B} (matroid groups). In the case of the embedding property we obtain the following.

PROPOSITION 1.3. *Let G be a group. Then G has the embedding property if and only if G has the pp-embedding property.*

Using the above proposition and Theorem 1.2 we also obtain the description of all finite solvable groups with the embedding property.

In Section 2 we present some properties of groups with the pp-embedding property and we prove Proposition 1.3. The main theorem (Theorem 1.2) is proved in Section 3.

2. Preliminary results

Our first observation concerns the Frattini quotient of G , i.e., the factor group of G by its Frattini subgroup. Throughout this text by a *Frattini-free group* we mean a group with trivial Frattini subgroup.

THEOREM 2.1. *Let G be a group and H be a normal subgroup of G satisfying $H \leq \Phi(G)$. Then G has the pp-embedding property if and only if G/H has the pp-embedding property.*

PROOF. Let $X \subseteq G$ be a set of pp-elements of G and \bar{G} denotes the quotient group G/H . By Lemma 4.1 of [7], we know that $x \in X$ is a pp-element of G if and only if \bar{x} is a pp-element of \bar{G} . Moreover, by 5.2.13 (c) of [11], we have $\Phi(\bar{G}) = \Phi(G)$. Hence, for every $x \in X$, $x \in \langle X \setminus \{x\}, \Phi(G) \rangle$ if and only if $\bar{x} \in \langle \bar{X} \setminus \{\bar{x}\}, \Phi(\bar{G}) \rangle$. It means that X is a pp-independent set of G if and only if \bar{X} is a pp-independent set of \bar{G} . From this, by straightforward calculations, we obtain that X can be embedded in a pp-base of G if and only if \bar{X} can be embedded in a pp-base of \bar{G} . \square

PROOF OF PROPOSITION 1.3. In view of Theorem 2.1, we can assume that G is Frattini-free. First we suppose that G has the embedding property. Let X be a pp-independent set of G . Then X is also a g-independent set of G . If $\langle X \rangle \neq G$ then, by assumption, there exists $y \in G$ such that $X \cup \{y\}$ is a g-independent set of G . If y is not a pp-element then, by Lemma 4.1 of [7], $y = y_1 \dots y_k$ where y_1, \dots, y_k are pp-elements of pairwise coprime orders. Moreover for each $x \in X$, we have

$$\langle X \setminus \{x\}, y_1, \dots, y_k \rangle = \langle X \setminus \{x\}, y \rangle < \langle X, y \rangle = \langle X, y_1, \dots, y_k \rangle.$$

It follows that there exist elements $y_{i_1}, \dots, y_{i_s} \subseteq \{y_1, \dots, y_k\}$ such that $X \cup \{y_{i_1}, \dots, y_{i_s}\}$ is pp-independent set of G and

$$\langle X, y_{i_1}, \dots, y_{i_s} \rangle = \langle X, y_1, \dots, y_k \rangle = \langle X, y \rangle.$$

If $\langle X, y_{i_1}, \dots, y_{i_s} \rangle \neq G$ then we can repeat the same argument as above to $\{X, y_{i_1}, \dots, y_{i_s}\}$.

Now suppose that G has the pp-embedding property. Let $X = \{g_1, \dots, g_k\}$ be a g-independent set of G . Again, by Lemma 4.1 of [7], there exists a pp-independent set $X' = \{g'_1, \dots, g'_s\}$ such that $\langle X' \rangle = \langle X \rangle$. Moreover for every $j \in \{1, \dots, k\}$ there exists in X' at least one power of g_j . By assumption, there exists a set $Y = \{g_{k+1}, \dots, g_n\}$ of pp-elements such that $X' \cup Y$ is a pp-base of G . Thus $\langle X \cup Y \rangle = G$. If $g_j \in \langle X \setminus \{g_j\}, Y \rangle$ for some $j \in \{1, \dots, k\}$ then the power of g_j could be omitted in $X' \cup Y$, a contradiction. So $X \cup Y$ is a g-base of G . \square

Based on the above theorem from now we will consider groups only with the pp-embedding property.

THEOREM 2.2. *Let G_1, G_2 be groups with coprime orders. If G_1 and G_2 have the pp-embedding property then $G_1 \times G_2$ has the pp-embedding property.*

PROOF. Assume that $G_1 \times G_2$ has the pp-embedding property. Consider G_1 and G_2 as contained in $G_1 \times G_2$ in a natural way. Observe that $x \in G_1 \times G_2$ is a pp-element if and only if $x \in G_1 \cup G_2$ is a pp-element.

Let X be a pp-independent set of $G_1 \times G_2$. Then $X = X_1 \cup X_2$, where $X_1 \subset G_1$ and $X_2 \subset G_2$ are sets of pp-elements. Obviously X_1 and X_2 are pp-independent sets of G_1 and G_2 respectively. By assumption, there exist sets $Y_1 \subset G_1$ and $Y_2 \subset G_2$ of pp-elements such that $X_1 \cup Y_1$ and $X_2 \cup Y_2$ are pp-bases of G_1 and G_2 respectively. Then $X_1 \cup X_2 \cup Y_1 \cup Y_2$ is a pp-base of $G_1 \times G_2$. \square

The following examples show that if G_1 and G_2 do not have coprime orders then $G_1 \times G_2$ does not necessarily have the pp-embedding property.

EXAMPLE 2.3. Let p, q, r be different odd primes. Set

$$\begin{aligned} G_1 &= \langle a, x \mid a^p = x^q = 1, a^x = a^i \rangle, \\ G_2 &= \langle b, y \mid b^q = y^2 = 1, b^y = b^{-1} \rangle, \\ G_3 &= \langle c, z \mid c^r = z^2 = 1, c^z = c^{-1} \rangle, \end{aligned}$$

where $q \mid (p-1)$ and $i^q \equiv 1 \pmod{p}$. Since these groups have rather simple structure, one check easily that they have the pp-embedding property.

Now we consider the group $G_1 \times G_2$. Observe that $o(bx) = q$ and so $X = \{a, b, bx\}$ is a pp-independent set of $G_1 \times G_2$. Moreover $\langle X \rangle \neq G_1 \times G_2$. Since $(bx)^y (bx)^{-1} = b$ and $y \notin \langle X \rangle$, X cannot be embedded in any pp-base of $G_1 \times G_2$. Hence $G_1 \times G_2$ does not have the pp-embedded property.

However in $G_2 \times G_3$ for every pp-element $g_1 \in G \setminus F(G_2 \times G_3)$, there exists pp-element $g_2 \in G \setminus \langle b, c, g_1 \rangle$ such that $[g_1, g_2] = 1$. Then $\{b, c, g_1, g_2\}$ is a pp-independent set of G . Using this it can be proved that $G_2 \times G_3$ has the pp-embedded property.

From the structures of finite nilpotent group and Theorem 2.5 of [10] we obtain the following.

PROPOSITION 2.4. *Every nilpotent group has the pp-embedding property.*

One may ask whether subgroups of a group with the pp-embedding property have the pp-embedding property. Now we will give an example showing that the answer is negative.

EXAMPLE 2.5. Let $p \neq q$ be primes such, that q is odd and $q|(p-1)$. Consider the group $G = P \rtimes Q$, where

$$P = \langle a, b \mid a^p = b^p = c^p = 1 = [a, c] = [b, c], c = [a, b] \rangle, \quad Q = \langle x \mid x^q = 1 \rangle$$

and $a^x = a^i, b^x = b^i$, with $i^q \equiv 1 \pmod{p}$. Thus $\Phi(G) = \Phi(P) = \langle c \rangle$ and $c^x = c^{i^2}$. It is easy to observe, that G has the pp-embedding property and property \mathcal{B} . However, if we take $H = \langle a, c, x \rangle$ then $\{ac, c\}$ is a pp-independent set of H , but it can not be embedded in any pp-base. Moreover as $\{ac, x\}$ is also a pp-base of H , H does not have property \mathcal{B} .

3. Solvable groups with pp-embedding property

In this chapter we investigate the finite solvable groups satisfying the pp-embedding property. It was noticed in [5] that the classification of finite solvable groups with the embedding property that is Theorem 1.2 of [13] is not correct. It was provided the following example.

EXAMPLE 3.1. Let G_1 and G_3 be as in Example 2.3. Then, by Theorem 2.2, $G_1 \times G_3$ has the pp-embedding property. However $|G_1 \times G_3 : F(G_1 \times G_3)| = 2q$. Hence $G_1 \times G_3$ does not satisfy conditions of Theorem 1.2 of [13].

Hence only one implication of Theorem 1.2 of [13] is correct namely:

THEOREM 3.2. *Let G be a group. If G is either abelian or Frattini-free group such that $F(G)$ has prime index in G and all its subgroups are normal in G , then G has the embedding property.*

In view of Theorem 2.1 we will restrict our consideration to Frattini-free groups. We begin our study of solvable groups with a simple but useful observation.

LEMMA 3.3. *Let G be a Frattini-free solvable group with the pp-embedding property and N be a subgroup of $F(G)$. Suppose that B is a pp-base of N and X is a pp-independent set of G with $N \cap \langle X \rangle = 1$. Then $B \cup X$ is a pp-independent set of G .*

PROOF. We observe that B is also a pp-independent set of G , as $\Phi(G) = \Phi(N) = 1$. Since, by Theorem 1.1, every subgroup of N is normal in G , for every $B' \subset B$

or every $X' \subset X$ we have

$$\begin{aligned} \langle B' \cup X \rangle &= \langle B' \rangle \langle X \rangle < \langle B \rangle \langle X \rangle = \langle B \cup X \rangle \quad \text{and} \\ \langle B \cup X' \rangle &= \langle B \rangle \langle X' \rangle < \langle B \rangle \langle X \rangle = \langle B \cup X \rangle. \end{aligned}$$

Hence $B \cup X$ is a pp-independent set of G . \square

LEMMA 3.4. *Let G be a Frattini-free solvable group and N be an abelian normal subgroup of G . If G has the pp-embedding property then G/N has the pp-embedding property.*

PROOF. Let N be an abelian normal subgroup of G . Since G is Frattini-free, there exists a subgroup K of G such that $G = N \rtimes K$, by [11, 5.2.13]. Let $\{g_1N, \dots, g_iN\}$ be a pp-independent set of G/N . Without loss of generality we may assume that $g_j \in K$ for $j = 1, \dots, i$ and hence $\langle g_1, \dots, g_i \rangle \cap N = 1$. By Lemma 4.1 of [7], we may further assume that g_j is a pp-element of G for $j \in \{1, \dots, i\}$. So, by straightforward calculation, we conclude that $\{g_1, \dots, g_i\}$ is a pp-independent set. Suppose that A is a pp-base of N . Then, by Lemma 3.3, $A \cup \{g_1, \dots, g_i\}$ is a pp-independent set of G . So, by assumption, there exist pp-elements $g_{i+1}, \dots, g_n \in G$ such that $A \cup \{g_1, \dots, g_n\}$ is a pp-base of G . It follows that $\langle g_1N, \dots, g_nN \rangle = G/N$. If for example $g_1N \in \langle g_2N, \dots, g_nN \rangle$, then $g_1 \in \langle A, g_2, \dots, g_n \rangle$, a contradiction. Hence $\{g_1N, \dots, g_nN\}$ is a pp-base of G/N . \square

THEOREM 3.5. *Every homomorphic image of a solvable group with the pp-embedding property has the pp-embedding property.*

PROOF. Let H be a normal subgroup of G . In view of Theorem 2.1, we may assume that $H \not\subseteq \Phi(G)$. Set $\bar{G} = G/\Phi(G)$. Then $\bar{H} \neq 1$ and $\bar{H} \triangleleft \bar{G}$. Let \bar{G} be a minimal counter-example. We assume that \bar{N} is a minimal normal subgroup of \bar{G} , such that \bar{N} is a subgroup of \bar{H} . Then, by Lemma 3.4, \bar{G}/\bar{N} has the pp-embedding property. It follows, by induction assumption, that \bar{G}/\bar{H} has the pp-embedding property as a homomorphic image of \bar{G}/\bar{N} .

Moreover there exists surjective homomorphism from \bar{G}/\bar{H} to $(G/H)/\Phi(G/H)$. Since $|\bar{G}/\bar{H}| < |\bar{G}|$, by the induction assumption, $(G/H)/\Phi(G/H)$ has the pp-embedding property. Again by Theorem 2.1, G/H has the pp-embedding property. \square

COROLLARY 3.6. *Let G_1 and G_2 be solvable groups. If $G_1 \times G_2$ has the pp-embedding property, then G_1 and G_2 have the pp-embedding property.*

THEOREM 3.7. *Let G be a Frattini-free solvable group. If G has the pp-embedding property then every pp-element of G has prime order.*

PROOF. Let G be as in Theorem 1.1 and $x \in G \setminus F(G)$ be a pp-element. Then $x = fh$, where $f \in F(G)$ and $1 \neq h \in H$. Assume that $1 \neq F(G) \cap \langle x \rangle = \langle a_1 \rangle$. Since $F(G)$ has the pp-embedding property, there exists $F \subset F(G)$ such that

$F \cup \{a_1\}$ is a pp-base of $F(G)$. From Theorem 1.1 follows that $\langle F \rangle \triangleleft G$. Moreover $\langle F(G), x \rangle / \langle F \rangle = \langle F, x \rangle / \langle F \rangle$ is cyclic. On the other hand $\langle F(G), x \rangle = \langle F(G), h \rangle = \langle F, a_1, h \rangle$ and so $\langle a_1, h \rangle$ is cyclic too. But this contradicts the choice of h . Hence $F(G) \cap \langle x \rangle = 1$.

Now suppose that x is a pp-element of order at least p^2 . Moreover let X be a pp-base of $F(G)$. By Lemma 3.3, $X \cup \{x^p\}$ is a pp-independent set of G . So, by assumption, there exist pp-elements x_1, \dots, x_k such that $X \cup \{x^p, x_1, \dots, x_k\}$ is a pp-base of G . Then $G/F(G) = \langle x^p F(G), x_1 F(G), \dots, x_k F(G) \rangle$. Since $G/F(G)$ is abelian, $x^p F(G) \in \Phi(G/F(G))$. It follows that $G/F(G) = \langle x_1 F(G), \dots, x_k F(G) \rangle$, so $G = \langle X, x_1, \dots, x_k \rangle$, a contradiction. Hence every pp-element of H has prime order. \square

LEMMA 3.8. *Let G be a Frattini-free group and G be a semidirect product $F(G) \rtimes \langle x \rangle$, where x is an element of prime order such that $C_{F(G)}(x) = 1$ and every subgroup of $F(G)$ is normal in G . Then*

- (1) *If $N \triangleleft G$, then $N \leq F(G)$ or $N = G$.*
- (2) *G is an indecomposable group.*
- (3) *G has the pp-embedding property.*

PROOF. (1) Suppose that N is a normal subgroup of G such that $N \not\subseteq F(G)$ and $F(G) \not\subseteq N$. By assumption the Frattini subgroup of $F(G)$ is also trivial, so $F(G)$ is a direct product of elementary abelian groups. In particular, G is solvable. It follows that there exists an element of prime order, say a , with $a \in F(G) \setminus N$. Indeed if all prime order elements of $F(G)$ are contained in N , then $F(G) \subseteq N$. Let $g \in N \setminus F(G)$. Since every subgroup of $F(G)$ is normal in G , $g^{-1}g^a \in \langle a \rangle$. However $\langle a \rangle \cap N = 1$ and so $g^a \notin N$, a contradiction. It follows that $F(G) \subset N$. Since $|G : F(G)|$ is a prime, $N = G$.

(2) If N_1, N_2 are proper normal subgroups of G satisfying $N_1 \cap N_2 = 1$, then by (1) $N_1 \times N_2 \leq F(G) \neq G$. Hence G is indecomposable.

(3) Follows from Theorem 3.2. \square

LEMMA 3.9. *Let G be a Frattini-free solvable group with the pp-embedding property. Then an indecomposable direct factor of G is one of the following:*

- (1) *a group of prime order;*
- (2) *a semidirect product $F(G_i) \rtimes \langle x \rangle$, where x is an element of prime order such that $C_{F(G_i)}(x) = 1$ and every subgroup of $F(G_i)$ is normal in G_i*

PROOF. Assume that G is not abelian. By Theorem 1.1, $G = F(G) \rtimes H$ and $G' \leq F(G)$. Let P_1 be a Sylow subgroup of $F(G)$ selected amongst those that are not direct factors of G . Then there exists a pp-element $x \in G \setminus F(G)$ such that $[x, P_1] \neq 1$. Otherwise $G = P_1 \times (F \rtimes H)$, where F is a complement of P_1 in $F(G)$. Moreover we assume that P_1, P_2, \dots, P_n are all Sylow subgroups of $F(G)$ such that $[P_i, x] \neq 1$. By Theorem 3.7, we know that x has prime order. Set $o(x) = q$, where q is a prime and $G_1 = (P_1 \times \dots \times P_n) \rtimes \langle x \rangle$.

Suppose that for some $i = 1, \dots, n$ there exist in P_i elements $a, b \neq 1$ such that $a^x = a$ and $b^x \neq b$. Since every subgroup of $F(G)$ is normal in G , we obtain $b^x = b^l$ for some $l \in \{2, \dots, p-1\}$. However then $(ab)^x = ab^l \notin \langle ab \rangle$, a contradiction. Hence x acts on $F(G_1)$ fixed-point-freely. It follows that $Z(G_1) = 1$.

Observe that $G_1 \triangleleft G$. Indeed since $G_1 \triangleleft (P_1 \times \dots \times P_n) \rtimes H$ and $\langle x \rangle$ centralizes a complement of $P_1 \times \dots \times P_n$ in $F(G)$, $G_1 \triangleleft G$.

Let $A = \{a_1, \dots, a_r\}$ be a pp-base of $P_1 \times \dots \times P_n$. Then, by Lemma 3.3, $\{a_1, \dots, a_r, x\}$ is a pp-independent set of G . Now we show that G_1 is a direct factor of G . In this purpose we consider a set $\{a_1x, \dots, a_rx, x\}$. From Theorem 10.5.1 of [11], we know that $a_i^{1+x^{-1}+\dots+x^{-(q-1)}} = 1$. Moreover $(a_ix)^q = a_i^{1+x^{-1}+\dots+x^{-(q-1)}}x^q = 1$. So $o(a_ix) = o(x) = q$ for every $i = 1, \dots, r$. If $a_ix \in \langle a_1x, \dots, a_{i-1}x, a_{i+1}x, \dots, a_rx, x \rangle$, for $i = 1, \dots, r$, then $a_i \in \langle a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r, x \rangle$, a contradiction. Moreover $\langle a_1a_r^{-1}, \dots, a_{r-1}a_r^{-1} \rangle < \langle a_1, \dots, a_r \rangle$. Hence we obtain

$$\begin{aligned} \langle a_1x, \dots, a_rx \rangle &= \langle a_1a_r^{-1}, \dots, a_{r-1}a_r^{-1}, a_rx \rangle = \langle a_1a_r^{-1}, \dots, a_{r-1}a_r^{-1} \rangle \langle a_rx \rangle < \\ &< \langle a_1, \dots, a_r \rangle \langle a_rx \rangle = \langle a_1x, \dots, a_rx, x \rangle. \end{aligned}$$

So $\{a_1x, \dots, a_rx, x\}$ is pp-independent in G . If $G_1 \neq G$, then there exists a pp-element $y \in G \setminus G_1$ such that $\{a_1x, \dots, a_rx, x, y\}$ is also pp-independent in G . Since G_1 is a normal subgroup of G , $[x, y] \in G_1 \cap F(G) = F(G_1)$.

Suppose that $[x, y] \neq 1$. Let $a \in \langle [x, y] \rangle$ be a pp-element. Then $a \in P_i$ for some $i = 1, \dots, n$. Since P_i is elementary abelian, P_i can be regarded as a vector space over some field of prime order. Hence there exists an pp-element, say $a_j \in A$ such that

$$\langle a_1, \dots, a_{j-1}, a, a_{j+1}, \dots, a_n \rangle = \langle a_1, \dots, a_n \rangle.$$

Thus

$$\langle a_1x, \dots, a_rx, x, y \rangle = \langle a_1x, \dots, a_{i-1}x, a_{i+1}x, a_rx, x, y \rangle,$$

a contradiction. Hence $[x, y] = 1$. Assume now that $[y, a_i] \neq 1$ for some $i = 1, \dots, r$. Then $[a_ix, y] = [a_i, y] \in \langle a_i \rangle$ and so $a_i \in \langle a_ix, y \rangle$. It follows $x \in \langle a_ix, y \rangle$, a contradiction. Hence $[G_1, y] = 1$.

Let $y_1, \dots, y_k \in G$ be pp-elements such that $\{a_1, \dots, a_r, x, y_1, \dots, y_k\}$ is a pp-base of G . Set $K = \langle y_1, \dots, y_k \rangle$. From above consideration we have that for every $i = 1, \dots, k$, $[y_i, G_1] = 1$. Hence $[g, G_1] = 1$, for every $g \in K$. Suppose that $1 \neq g \in G_1 \cap K$. Then $g \in Z(G_1)$, but this contradicts the fact that $Z(G_1) = 1$. It follows that $G = G_1 \times K$.

By Lemma 3.8 we know that G_1 is an indecomposable direct factor. Since Sylow subgroups of $F(K)$ are Sylow subgroups of $F(G)$, by analogous consideration we obtain that $G = G_1 \times \dots \times G_n$, where G_i is an indecomposable direct factor of G occurring in (1) and (2) of the statement of the lemma. If $G = H_1 \times \dots \times H_k$ is also a direct decomposition of G on indecomposable factors, then in view of Krull-Remak-Schmidt Theorem, $n = k$ and for every $j \in \{1, \dots, n\}$ there exists $i \in \{1, \dots, n\}$ such that $H_j \cong G_i$. So the proof is complete. \square

PROOF OF THEOREM 1.2 Suppose first that G is a Frattini-free group with the pp-embedding property. If G is abelian, then $G = F(G)$ and the conclusion is obvious. So let G be a nonabelian group. Hence, by Krull-Remak-Schmidt Theorem, $G = G_1 \times \dots \times G_n$, where G_i is an indecomposable direct factor for $i = 1, \dots, n$. Moreover all these factors are described in Lemma 3.9. We can assume that at least one of these direct factors, say G_1 , is nonabelian. It follows that there exists $x \in G_1$ such that $G_1 = F(G_1) \rtimes \langle x \rangle$. Hence it remains to show that $(|F(G_1)|, |G_i|) = 1$, for $i \neq 1$.

In this purpose first assume that G_i is abelian. Since G_i is a Sylow subgroup of $F(G)$ and $F(G_1)$ is a direct product of Sylow subgroups of $F(G)$, $(|F(G_1)|, |G_i|) = 1$.

Hence suppose that G_i is nonabelian. Thus there exists $y \in G_i$ such that $G_i = F(G_i) \rtimes \langle y \rangle$. Since $F(G_1)$ and $F(G_i)$ are direct products of Sylow subgroups of $F(G)$, $(|F(G_1)|, |F(G_i)|) = 1$. By Theorem 3.5, $G_1 \times G_i$ has the pp-embedding property, as a homomorphic image of G . Let $\{a_1, \dots, a_r\}$ and $\{b_1, \dots, b_s\}$ be pp-bases of $F(G_1)$ and $F(G_i)$ respectively. Assume that $o(a_1) = o(y)$ is a prime number. By Lemma 3.3, $\{a_1, \dots, a_r, b_1, \dots, b_s, a_1 y\}$ is a pp-independent set of $G_1 \times G_i$. Then there exist pp-elements z_1, \dots, z_t such that $\{a_1, \dots, a_r, b_1, \dots, b_s, a_1 y, z_1, \dots, z_t\}$ is a pp-base of $G_1 \times G_i$. If for all $i = 1, \dots, t$, $[a_1, z_i] = 1$, then $a_1 \in Z(G_1 \times G_i)$, to contrary the fact that $Z(G_1 \times G_i) = 1$. Hence there exists z_i such that $[a_1, z_i] \neq 1$. It follows that $[a_1 y, z_i] = [a_1, z_i] \in \langle a_1 \rangle$. This is a contradiction. Hence $(|F(G_1)|, |G_i|) = 1$.

Conversely suppose that G has the structure given in the theorem. By Proposition 2.4, an abelian group has the pp-embedding property.

So we assume that there is a nonabelian direct factor among G_1, \dots, G_n . By induction on n we can assume that $K = G_1 \times \dots \times G_{n-1}$ has the pp-embedding property. Moreover by Lemma 3.8 and Proposition 2.4, G_n has the pp-embedding property. If $(|K|, |G_n|) = 1$, then by Theorem 2.2 we know that $K \times G_n$ has this property.

Hence suppose that $(|K|, |G_n|) \neq 1$. By the description of G in the theorem, it follows that $(|K|, |G_n|) = q$, for some prime q . Moreover for all $i, j \in \{1, \dots, n\}$ and $i \neq j$, either $(|G_i|, |G_j|) = 1$ or $(|G_i|, |G_j|) = q$. So, by Theorem 2.2, without loss of generality we can assume that $q \mid |G_i|$ for $i = 1, \dots, n$. Let $X = \{x_1, \dots, x_m\}$ be a pp-independence set of G and $N = \langle X \rangle \neq G$.

If $F(G) \not\leq N$, then there exists a pp-element $y \in F(G) \setminus N$. Since $\langle y \rangle$ is a normal subgroup of G , for every $j = 1, \dots, m$ we obtain

$$\langle X \setminus \{x_j\}, y \rangle = \langle X \setminus \{x_j\} \rangle \langle y \rangle < \langle X \rangle \langle y \rangle.$$

Hence $X \cup \{y\}$ is a pp-independent set of G .

Thus assume that $F(G) \leq N$. Set $x_j = (x_{j1}, x_{j2})$ where $x_{j1} \in K$ and $x_{j2} \in G_n$ for $j = 1, \dots, m$. We have to consider two cases:

- (i) $\langle x_{11}, \dots, x_{m1} \rangle \neq K$ or $\langle x_{12}, \dots, x_{m2} \rangle \neq G_n$,
- (ii) $\langle x_{11}, \dots, x_{m1} \rangle = K$ and $\langle x_{12}, \dots, x_{m2} \rangle = G_n$.

Case (i): Let $\langle x_{11}, \dots, x_{m1} \rangle \neq K$ and $X_1 \subseteq \{x_{11}, \dots, x_{m1}\}$ be pp-independent

in K . Then there exists $y \in K \setminus \langle x_{11}, \dots, x_{m1} \rangle$ such that $X_1 \cup \{y\}$ is pp-independent in K . It follows that $X \cup \{(y, 1)\}$ is a pp-independent in G .

Case (ii): Let $\langle x_{11}, \dots, x_{m1} \rangle = K$ and $\langle x_{12}, \dots, x_{m2} \rangle = G_n$. Then there exists some pp-element of X , say x_m , such that $x_{m1} \notin N$ and $x_{m2} \notin N$. Otherwise $N = G$. It follows that $o(x_m) = q$ and further $o(x_{1m}) = o(x_{m2}) = q$. If G_n is abelian then $G_n \leq F(K) \times F(G_n) \leq N$. Hence $\langle x_{11}, \dots, x_{m1} \rangle \neq K$ and we have case (i). So G_n is nonabelian. Thus $x_{m1} \in K \setminus F(K)$ and $x_{m2} \in G_n \setminus F(G_n)$.

Set $y = (x_{m1}, 1)$. We show that $X \cup \{y\}$ is a pp-independent set of G . If $F(G) \leq \langle X \setminus \{x_j\} \rangle$, then $\langle X \setminus \{x_j\} \rangle \triangleleft G$ and

$$\langle X \setminus \{x_j\}, y \rangle = \langle X \setminus \{x_j\} \rangle \langle y \rangle < \langle X \rangle \langle y \rangle = \langle X, y \rangle.$$

If $F(G) \not\leq \langle X \setminus \{x_j\} \rangle$, then

$$[x_k, y] = ([x_{k1}, x_{m1}], 1) \quad \text{and} \quad [x_k, x_m] = ([x_{k1}, x_{m1}], [x_{k2}, x_{m2}]).$$

Since $[x_k, x_m] \in F(K) \times F(G_n)$ and $(|F(K)|, |F(G_n)|) = 1$, $[x_k, y] \in \langle [x_k, x_m] \rangle$. It follows that $F(G) \not\leq \langle X \setminus \{x_j\}, y \rangle$. As $F(G) \leq \langle X, y \rangle$, we obtain $\langle X \setminus \{x_j\}, y \rangle < \langle X, y \rangle$. So $X \cup \{y\}$ is a pp-independent set of G , and the proof is complete. \square

REMARK 3.10. It was proved in [10] that simple groups do not satisfy property \mathcal{B} . By [6], we know that every simple group is generated by an involution and element of prime order. On the other hand, by the Classification of Finite Simple Group, we know that every simple group is generated by at least three involutions. Hence all simple groups do not have property \mathcal{B}_{pp} . In [13] the authors conjecture that simple groups do not have the embedding property. However so far we do not know any proof of this fact.

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