Projection of a nonsingular plane quintic curve and the dihedral group of order eight

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Abstract — Let $C$ be a nonsingular plane quintic curve over the complex number field $\mathbb{C}$, and let $\pi_P : C \rightarrow \mathbb{P}^1$ be a projection from $P \in C$. Let $L_P$ be the Galois closure of the field extension $\mathbb{C}(C)/\mathbb{C}(\mathbb{P}^1)$ induced by $\pi_P$, where $\mathbb{C}(C)$ and $\mathbb{C}(\mathbb{P}^1)$ are the rational function fields of $C$ and $\mathbb{P}^1$, respectively. We call the point $P$ a $D_4$-point if the Galois group of $L_P/\mathbb{C}(\mathbb{P}^1)$ is isomorphic to the dihedral group $D_4$ of order eight.

In this paper, we prove that the number of $D_4$-points for $C$ equals 0, 1, 3, 5, or 15, and show that the curve with 15 $D_4$-points is projectively equivalent to the Fermat quintic curve.

1. Introduction

We shall work over the complex number field $\mathbb{C}$. Let $C \subset \mathbb{P}^2$ be a nonsingular plane curve of degree $d \geq 2$, and let $\pi_P : C \rightarrow \mathbb{P}^1$ be the projection from a point $P \in \mathbb{P}^2$. The projection $\pi_P$ induces an extension $\pi_P^* : \mathbb{C}(\mathbb{P}^1) \hookrightarrow \mathbb{C}(C)$, where $\mathbb{C}(C)$ and $\mathbb{C}(\mathbb{P}^1)$ are the rational function fields of $C$ and $\mathbb{P}^1$, respectively. We denote by $K$ and $K_P$ the function field $\mathbb{C}(C)$ and its subfield $\pi_P^*(\mathbb{C}(\mathbb{P}^1))$, respectively. Let $L_P$ be the Galois closure of $K/K_P$, and let $\text{Gal}(L_P/K_P)$ be the Galois group of the field extension $L_P/K_P$.

The study of the projections $\pi_P$ is an interesting issue of nonsingular plane curves. Indeed, a classical theorem of Noether and later results (see e.g. [5]) assure that the minimum degree of a morphism $C \rightarrow \mathbb{P}^1$ equals $d-1$, and all the maps of degree $d-1 \geq 2$ (resp. $d \geq 5$) are projections $\pi_P : C \rightarrow \mathbb{P}^1$ from some point $P \in C$ (resp. $P \in \mathbb{P}^2 \setminus C$). Thus it is natural to investigate the Galois group $\text{Gal}(L_P/K_P)$ associated to projections $\pi_P$.

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Cukierman [1] has shown that if \( C \) is a general plane curve of degree \( d \), then for every point \( P \in \mathbb{P}^2 \setminus C \), the Galois group \( \text{Gal}(L_P/K_P) \) is isomorphic to the symmetric group on \( d \) letters. Without any assumption of generality, Miura and Yoshihara [3, 8] have shown that if \( P \in C \setminus C^\# \) (resp. \( P \in \mathbb{P}^2 \setminus (C \cup C^\#) \)), then the Galois group \( \text{Gal}(L_P/K_P) \) is isomorphic to the symmetric group on \( d-1 \) letters (resp. on \( d \) letters), where \( C^\# \) is the union of all multitangent lines to \( C \). Moreover, Pirola and Schlesinger [7] have shown that there are only finitely many points \( P \in \mathbb{P}^2 \setminus C \) for which the Galois group \( \text{Gal}(L_P/K_P) \) is not isomorphic to the symmetric group on \( d \) letters.

In [3, 8], the notion of “Galois point” has been introduced: a point \( P \) is said to be a Galois point if the extension \( K=K_P \) is Galois. Miura and Yoshihara have shown that if \( P \in C \) (resp. \( P \in \mathbb{P}^2 \setminus C \)) is a Galois point, then the Galois group \( \text{Gal}(K/K_P) \) is isomorphic to the cyclic group of order \( d-1 \) (resp. \( d \)). Furthermore, they have determined the number and distribution of Galois points.

As an extension of these studies, we would like to treat the case in which the Galois group \( L_P/K_P \) is not isomorphic to a cyclic group or a full symmetric group. In this paper, we study points \( P \in C \) lying on a nonsingular plane curve \( C \) of degree \( d = 5 \), such that the Galois group \( \text{Gal}(L_P/K_P) \) of the projections \( \pi_P : C \to \mathbb{P}^1 \) is isomorphic to the dihedral group \( D_4 \) of order 8.

**Definition 1.1.** A point \( P \in C \) is called a \( D_4 \)-point if the Galois group of \( L_P/K_P \) is isomorphic to the dihedral group \( D_4 \) of order 8. We denote the set of all \( D_4 \)-points as \( \Delta(C; D_4) \), and \( \delta(C, D_4) \) is the number of elements in this set, i.e.,

\[
\Delta(C, D_4) = \{ P \in C \mid P \text{ is a } D_4\text{-point} \}, \quad \delta(C, D_4) = \# \Delta(C, D_4).
\]

Our main theorem is as follows.

**Theorem 1.2.** Let \( C \) be a nonsingular plane quintic curve. Then, \( \delta(C, D_4) \in \{0, 1, 3, 5, 15\} \). Moreover,

1. If \( \delta(C, D_4) = 3 \) or 5, then all \( D_4 \)-points are collinear;
2. Equality \( \delta(C, D_4) = 15 \) holds if and only if \( C \) is projectively equivalent to the Fermat quintic curve; in this case \( C \) possesses three disjoint 5-tuples of collinear \( D_4 \)-points.

Furthermore, we present explicit examples showing that all the values of \( \delta(C, D_4) \) listed above occur.

In order to prove Theorem 1.2, the main techniques rely on the description of \( D_4 \)-points presented in [2]. In particular, since the projection \( \pi_P : C \to \mathbb{P}^1 \) from a \( D_4 \)-point factors through a double covering \( C \to C' \) of a genus two curve \( C' \), we deduce that each \( D_4 \)-point \( P \in C \) induces an involution \( \iota_P \in \text{Aut}(C) \). Then we achieve Theorem 1.2 by studying the action of the subgroup of \( \text{Aut}(C) \) generated by those involutions on the points of \( C \).

It is worth noting that, in the light of [2, 8] and Theorem 1.2, the only case to discuss for completing the description of projections from points on plane quintic
curves is when the Galois group is isomorphic to the alternating group $A_4$. Unfortunately, our techniques do not apply as projections from $A_4$-points do not factor through intermediate curves.

The plan of the paper is the following. In Section 2, we give some preliminary results on $D_4$-points for a nonsingular plane quintic curve. Section 3 concerns examples assuring that the numbers of $D_4$-points listed in Theorem 1.2 occur. In Section 4, we prove Theorem 1.2.

2. Preliminary Results

In this section, we are aimed at presenting the preliminary results necessary to study $D_4$-points.

Hereafter, $C \subset \mathbb{P}^2$ is a nonsingular curve of degree $d = 5$, and we use the following notation.

**Notation.** We denote by $\text{Aut}(C)$ the group of automorphisms of $C$, and by $\text{id}_C$ the identity automorphism. Consider a point $P \in C$ and the Galois closure $L_P$ associated to the projection $\pi_P : C \to \mathbb{P}^1$. Then, $\tilde{C}_P$ is the nonsingular projective curve having $L_P$ as rational function field, and $g(P)$ denotes its genus. We denote by $T_P C$ the tangent line to $C$ at $P$. Given two plane curves $A$ and $B$, let $I_P(A, B)$ denote their intersection multiplicity at $P$. Let $(X : Y : Z)$ be the homogeneous coordinates of the projective plane $\mathbb{P}^2$. Then we denote by $f_i(X, Y)$ any homogeneous polynomial of degree $i$. Given a homogeneous polynomial $f \in \mathbb{C}[X, Y, Z]$, we denote by $V(f)$ the plane curve defined by $f = 0$, and by $F(5) := V(X^5 + Y^5 + Z^5)$ the Fermat quintic curve. By $\text{Aut}(\mathbb{P}^2)$ we mean the group of projective linear transformation of $\mathbb{P}^2$, and for any $T \in \text{Aut}(\mathbb{P}^2)$ we define its fixed locus $\text{Fix}(T) := \{P \in \mathbb{P}^2 | T(P) = P\}$. Finally, we denote by $\#S$ the number of elements of a finite set $S$.

The following theorem summarizes Miura’s results on $D_4$-points (see [2]).

**Theorem 2.1.** Let $C$ be a nonsingular plane quintic curve. Then:

1. A point $P \in C$ is a $D_4$-point if and only if $P$ and the defining equation of $C$ can be expressed as

   $$P = (0 : 0 : 1) \text{ and } f_1(X, Y)Z^4 + f_3(X, Y)Z^2 + f_5(X, Y) = 0$$

   by taking a suitable projective transformation, where $f_i(X, Y) \neq 0$ ($i = 1, 3, 5$).

2. If $P \in C$ is a $D_4$-point, then there exists an intermediate field $K'$ of the extension $K/K_P$ such that the degrees $[K : K'] = 2$ and $[K' : K_P] = 2$. Moreover, the nonsingular projective curve whose function field is isomorphic to $K'$ has genus 2.
(3) If $P \in C$ is a $D_4$-point, then $g(P)$ equals 16 or 17. Let $\delta_{16}(C)$ (resp. $\delta_{17}(C)$) be the number of $D_4$-points satisfying $g(P) = 16$ (resp. $g(P) = 17$). Then, $5\delta_{16}(C) + 6\delta_{17}(C) \leq B(C)$, where $B(C)$ is the number of lines bitangent to $C$.

**Remark 2.2.** Every automorphism of a nonsingular plane curve $C$ of degree $d \geq 4$ can be extended to a projective transformation of $\mathbb{P}^2$ ([6, Corollary 5.3.19]), i.e., for every $\sigma \in \text{Aut}(C)$, there exists $T \in \text{Aut}(\mathbb{P}^2)$ such that $T|_C = \sigma$. Note that if $T \neq T'$, then $T|_C \neq T'|_C$. Indeed, if $T|_C = T'|_C$, then $\text{Fix}(T \circ T'^{-1}) \supset C$. Because $\text{Fix}(T \circ T'^{-1})$ is the union of some linear spaces, we have that $\text{Fix}(T \circ T'^{-1}) = \mathbb{P}^2$. Hence, $T \circ T'^{-1}$ is the identity, and so $T = T'$. In this paper, we shall often express $\sigma \in \text{Aut}(C)$ as an element of $\text{Aut}(\mathbb{P}^2)$, and we shall represent it by its representation matrix in $\text{PGL}(3, \mathbb{C})$. We note that an element $A \in \text{PGL}(3, \mathbb{C})$ induces a projective transformation $(\tilde{X} : \tilde{Y} : \tilde{Z}) \mapsto (X : Y : Z)$ so that $(X, Y, Z)^{tr} = A(X, Y, Z)^{tr}$, where $(X, Y, Z)^{tr}$ and $(X, Y, Z)^{tr}$ are transposed matrices of $(X, Y, Z)$ and $(X, Y, Z)$, respectively.

**Remark 2.3.** For an element $\sigma \in \text{Aut}(\mathbb{P}^2)$, let $A$ be its representation matrix. Then, a point $Q = (a : b : c)$ is a fixed point of $\sigma$, i.e., $\sigma(Q) = Q$, if and only if the vector $(a, b, c)$ is an eigenvector of $A$. Hence, $\text{Fix}(\sigma)$ is one of the following:

1. $\text{Fix}(\sigma)$ consists of one point, two points, or one line;
2. $\text{Fix}(\sigma)$ consists of three non-collinear points;
3. $\text{Fix}(\sigma)$ consists of one point and one line;
4. $\text{Fix}(\sigma) = \mathbb{P}^2$.

In particular, if the order of $\sigma$ is finite, then $\text{Fix}(\sigma)$ is of type (2), (3), or (4), because $A$ is diagonalizable.

By Theorem 2.1, we have the following lemmas.

**Lemma 2.4.** If $P \in C$ is a $D_4$-point for $C$, then $I_P(C, T PC) = 3$ or 5.

**Proof.** By Theorem 2.1 (1), we may assume that $P = (0 : 0 : 1)$ and the defining equation of $C$ is $f_1(X, Y)Z^2 + f_3(X, Y)Z^2 + f_5(X, Y) = 0$. Then, $T_PC$ is given by $f_1(X, Y) = 0$. Hence, we have the conclusion. \qed

**Lemma 2.5.** If $P \in C$ is a $D_4$-point for $C$, then there exists a unique involution $i_P \in \text{Aut}(C)$ such that

1. $i_P \neq \text{id}_C$ and it extends to an involution of $\mathbb{P}^2$, that is $i_P^2 = \text{id}_{\mathbb{P}^2}$;
2. $i_P(P) = P$ and $i_P(\ell) = \ell$ for every line $\ell$ passing through the point $P$;
3. $i_P$ has fixed locus $\text{Fix}(i_P) = \{P\} \cup \ell_P$, where $\ell_P$ is a line not passing through $P$. 


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Proof. By Theorem 2.1 (1), we may assume that \( P = (0 : 0 : 1) \) and the defining equation of \( C \) is \( f_1(X, Y)Z^4 + f_3(X, Y)Z^2 + f_5(X, Y) = 0 \). Then, we put
\[
\iota_P : (X : Y : Z) \mapsto (-X : -Y : Z).
\]
Hence \( \iota_P \in \text{Aut}(C) \) is an involution satisfying properties (1), (2) and (3). In particular, the line \( \ell_P \subset \text{Fix}(\iota_P) \) has equation \( Z = 0 \) and it does not pass through \( P \).

To check that \( \iota_P \) is unique, we assume that there exists another involution \( \iota'_P \in \text{Aut}(C) \) satisfying the same properties. Then, let \( G \) be the group generated by \( \iota_P \) and \( \iota'_P \). Both \( \iota_P \) and \( \iota'_P \) correspond to automorphisms of \( K \) that fix every element of \( K \). Hence, we see that the order of \( G \) satisfies \( \#G \leq [K : K_P] = 4 \). As \( K/K_P \) is not Galois, we have \( \#G = 2 \), and therefore, \( \iota_P = \iota'_P \).

We call \( \iota_P \) the involution associated to the \( D_4 \)-point \( P \), and \( \ell_P \) the line of fixed points of \( \iota_P \).

Remark 2.6. It follows from Theorem 2.1 (2) that the projection \( \pi_P : C \to \mathbb{P}^1 \) factors through a double covering \( \varphi_P : C \to C' \) and a hyperelliptic map \( \psi_P : C' \to \mathbb{P}^1 \), where \( C' \) is a curve of genus 2. Then it is easy to check that \( \iota_P \) is the involution associated to the map \( \varphi_P \), sending \( Q \in C \) to the other point \( Q' \in C \) such that \( \varphi_P(Q') = \varphi_P(Q) \). In particular, \( \iota_P \) fixes 6 points on \( C \) (\( P \) and \( C \cap \ell_P \)), and they coincide with the ramification points of \( \varphi_P \).

Lemma 2.7. Assume that \( P := (0 : 0 : 1) \in C \) and
\[
\iota := \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix} \in \text{Aut}(C).
\]
Then, the defining equation of \( C \) must be expressed as \( f_1(X, Y)Z^4 + f_3(X, Y)Z^2 + f_5(X, Y) = 0 \). In particular, \( P \) is a Galois point when \( f_3(X, Y) = 0 \), and \( P \) is a \( D_4 \)-point otherwise.

Proof. Let the defining equation of \( C \) be
\[
F(X, Y, Z) := \sum_{i=0}^{5} f_i(X, Y)Z^{5-i} = 0.
\]
Because \( P \in C \), \( P \) is not a singular point, and \( C \) is irreducible, we have that \( f_0(X, Y) = 0 \), \( f_1(X, Y) \neq 0 \) and \( f_5(X, Y) \neq 0 \). As \( \iota \in \text{Aut}(C) \), we have that \( \iota^*F = \lambda F \) for some \( \lambda \in \mathbb{C} \setminus \{0\} \). Hence,
\[
-f_1(X, Y)Z^4 + f_3(X, Y)Z^2 - f_5(X, Y)Z - f_5(X, Y) = \lambda F(X, Y, Z),
\]
so that \( \lambda = -1 \) and \( f_3(X, Y) = f_4(X, Y) = 0 \). By [2, Proposition 3.6] (resp. Theorem 2.1), if \( f_3(X, Y) = 0 \) (resp. \( f_5(X, Y) \neq 0 \)), then \( P \) is a Galois point (resp. a \( D_4 \)-point).
Lemma 2.8. Let $P$ and $Q$ be two $D_4$-points for $C$. Then, $\iota_P(Q) \neq P, Q$ is a $D_4$-point, and the three points $P$, $Q$, and $\iota_P(Q)$ are collinear.

Proof. By Theorem 2.1 and Lemma 2.5, we may assume that $P = (0 : 0 : 1)$, $C : f_1(X, Y)Z^4 + f_2(X, Y)Z^2 + f_3(X, Y) = 0$ and $\ell_P : Z = 0$. Because $\iota_P(P) = P$, we have that $\iota_P(Q) \neq P$. Assume that $\iota_P(Q) = Q$. Then $Q \in \ell_P$, and so $f_3(Q) = 0$. Therefore, the tangent line $T_QC$ has equation

$$\frac{\partial f_3}{\partial X}(Q)X + \frac{\partial f_3}{\partial Y}(Q)Y = 0,$$

and hence $P \in T_QC$. Thus $I_Q(C, T_QC) = 2$ or $4$, which contradicts Lemma 2.4. Therefore $\iota_P(Q) \neq Q$.

By Lemma 2.5, we see that the three points $P$, $Q$, and $\iota_P(Q)$ are collinear. \hfill \square

Lemma 2.9. For every line $\ell$, $\#(\Delta(C, D_4) \cap \ell) = 0, 1, 3,$ or $5$.

Proof. Assume that there exist two $D_4$-points $P_1$ and $P_2$ on $\ell$. Then, by Lemma 2.8, we obtain a third $D_4$-point $\iota_{P_1}(P_2)$ on $\ell$.

Assume that there exist four $D_4$-points $P_i (i = 1, 2, 3, 4)$ on $\ell$. By Lemma 2.8, we may assume that $\iota_{P_1}(P_2) = P_3$. Then, we see that $\iota_{P_1}(P_4) \neq P_i (i = 1, 2, 3, 4)$. Hence, we have a fifth $D_4$-point $\iota_{P_1}(P_4)$ on $\ell$. \hfill \square

Lemma 2.10. Let $P_1$ and $P_2$ be two $D_4$-points for $C$, and let $\ell$ be the line passing through the points $P_1$ and $P_2$. If $\#(\Delta(C, D_4) \cap \ell) = 3$ (resp. $5$), then the order of $\iota_{P_1}|_\ell \circ \iota_{P_2}|_\ell$ equals $3$ (resp. $5$).

Proof. By Lemma 2.8, we have the third $D_4$-point $P_3 := \iota_{P_1}(P_2)$. With a suitable projective transformation, we may assume that $P_1 = (0 : 0 : 1)$, $P_2 = (0 : 1 : 0)$, $P_3 = (0 : 1 : 1)$, and $\ell : X = 0$. From Lemmas 2.4 and 2.8, we see that $\ell$ is not a tangent line to $C$ at $P_i$, where $i = 1, 2, 3$. Indeed, if $\ell$ were the tangent line to $C$ at $P_i$, then $\ell$ would be also tangent at $\iota_{P_i}(P_i)$, where $j \neq i$. Thus the intersection number would be $I(C, \ell) > 5$, which is a contradiction. Hence, we have that $\#(C \cap \ell) = 4$ or $5$. Let $P_4 := (0 : \alpha : 1) \in (C \cap \ell) \setminus \{P_1, P_2, P_3\}$ and $P_5 := \iota_{P_4}(P_4)$, where $\alpha \in C \setminus \{0, 1\}$. Note that if $\#(C \cap \ell) = 4$, then $P_4 = P_5$. From $\iota_{P_1}(P_1) = P_1$, $\iota_{P_1}(P_2) = P_3$, and $\iota_{P_1}(P_3) = P_2$, we infer that

$$\iota_{P_1}|_\ell = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \in \text{PGL}(2, \mathbb{C})$$

and $P_5 = (0 : \alpha : \alpha - 1)$. Here, we have two cases, $\iota_{P_2}(P_1) = P_5$ or $\iota_{P_2}(P_1) \in \{P_4, P_5\}$.

If $\iota_{P_2}(P_1) = P_5$, we can infer from $\iota_{P_2}(P_2) = P_2$, $\iota_{P_2}(P_1) = P_5$, $\iota_{P_2}(P_3) = P_1$, and $\iota_{P_2}(P_4) = P_5$ that

$$\iota_{P_2}|_\ell = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{PGL}(2, \mathbb{C})$$
and \( \alpha^2 - \alpha + 1 = 0 \). Hence, we may assume that \( P_4 = (0 : -\omega : 1) \) and \( P_5 = (0 : 1 : -\omega) \), where \( \omega \) is a primitive cubic root of unity. We see that the order of \( \iota_{P_4} | \ell \circ \iota_{P_4} | \ell \) equals 3. If \( P_4 \) is a \( D_4 \)-point, then there exists the involution \( \iota_{P_4} \) such that \( \iota_{P_4}(P_4) = P_4 \), \( \iota_{P_4} \) acts on \( \{ P_1, P_2, P_3, P_5 \} \), and \( \iota_{P_4}^2 = \text{id}_C \). However, we see that there does not exist such an element \( \iota_{4} | \ell \) in \( \text{PGL}(2, \mathbb{C}) \). Therefore, \( \#(\Delta(C, D_4) \cap \ell) = 3 \).

If \( \iota_{P_3}(P_1) \in \{ P_4, P_5 \} \), then because \( \iota_{P_4} \circ \iota_{P_3} \) acts transitively on the five points \( P_1, \ldots, P_5 \), we have that the order of \( \iota_{P_3} | \ell \circ \iota_{P_3} | \ell \) equals 5, and both \( P_4 \) and \( P_5 \) are \( D_4 \)-points.

**Remark 2.11.** By the proof of Lemma 2.10, if \( \#(\Delta(C, D_4) \cap \ell) \geq 2 \) for a line \( \ell \), then \( \#(C \cap \ell) = 5 \).

**Proposition 2.12.** Let \( P \) and \( Q \) be two \( D_4 \)-points for \( C \). Then, there exists an automorphism \( \sigma \in \text{Aut}(C) \) such that \( \sigma(P) = Q \).

**Proof.** Let \( R := \iota_P(Q) \), so that \( P, Q \) and \( R \) are distinct collinear points by Lemma 2.8. If \( \iota_Q(R) = P \), then \( \iota_P \circ \iota_Q(P) = Q \). If \( \iota_Q(R) \neq P \), then let \( R' := \iota_Q(R) \) and \( R'' := \iota_Q(P) \). In particular, \( P, Q, R, R' \) and \( R'' \) are collinear \( D_4 \)-points, and hence \( \iota_P(R') = R'' \). Thus we have that \( \iota_P \circ \iota_Q \circ \iota_P \circ \iota_Q(P) = Q \).

By Proposition 2.12, if there exist two \( D_4 \)-points \( P \) and \( Q \) for \( C \), then \( \bar{C}_P \) is isomorphic to \( \bar{C}_Q \), and in particular \( g(P) = g(Q) \).

We can improve the inequality stated in Theorem 2.1 (3) as follows.

**Lemma 2.13.** Let \( P \) be a \( D_4 \)-point for \( C \). Then, we have \( A_2 + A_3 + B_1 = 6 \), where \( A_2 \) is the number of lines that intersect \( C \) transversally at \( P \) and meet \( C \) with multiplicity 4 at another point, \( A_3 \) is the number of lines that meet \( C \) with multiplicity 5 at \( P \), and \( B_1 \) is the number of lines that intersect \( C \) transversally at \( P \) and are tangent to \( C \) at two other distinct points.

**Proof.** Let \( \ell \) be a line passing through \( P \). By Lemma 2.4 we have that \( I_P(C, \ell) \in \{ 1, 3, 5 \} \). Moreover, Lemma 2.5 assures that \( \ell \neq \ell_P \), and that \( \iota_P \) permutes points on \( (C \cap \ell) \setminus \text{Fix}(\iota_P) \). In particular, for any \( Q \in (C \cap \ell) \setminus \text{Fix}(\iota_P) \) we have that \( I_Q(C, \ell) = I_{\iota_P(Q)}(C, \ell) \). Thus only the following cases may occur:

1. The line \( \ell \) intersects \( C \) transversally at \( P \) and four other distinct points that are not on \( \ell_P \);
2. The line \( \ell \) intersects \( C \) transversally at \( P \) and two other distinct points, and it is tangent to \( C \) at another point lying on \( \ell_P \); (Let \( A_0 \) be the number of these lines.)
3. The line \( \ell \) is tangent to \( C \) at \( P \) with multiplicity 3, and intersects \( C \) transversally at two other distinct points that are not on \( \ell_P \); (Let \( A_1 \) be the number of these lines.)
(4) The line \( \ell \) intersects \( C \) transversally at \( P \), and is tangent to \( C \) with multiplicity 4 at another point lying on \( \ell_P \); (Let \( A_2 \) be the number of these lines.)

(5) The line \( \ell \) is tangent to \( C \) at \( P \) with multiplicity 5; (Let \( A_3 \) be the number of these lines.)

(6) The line \( \ell \) intersects \( C \) transversally at \( P \), and is tangent to \( C \) at two other distinct points that are not on \( \ell_P \); (Let \( B_1 \) be the number of these lines.)

(7) The line \( \ell \) is tangent to \( C \) at \( P \) with multiplicity 3, and is tangent to \( C \) at another point lying on \( \ell_P \). (Let \( B_2 \) be the number of these lines.)

Consider the morphism \( \pi_P : C \to \mathbb{P}^1 \), where the genus \( g(C) = 6 \) and \( \deg \pi_P = 4 \). Then, by the Riemann–Hurwitz formula \( 2g(C) - 2 = (\deg \pi_P)(2g(\mathbb{P}^1) - 2) + \deg R \), where \( R \) is the ramification divisor, we have that \( \deg R = A_0 + A_1 + 3A_2 + 3A_3 + 2B_1 + 2B_2 = 18 \). Since the tangent line to \( C \) at \( P \) is unique, we have \( A_1 + A_3 + B_2 = 1 \). Since \( \# (C \cap \ell_P) = 5 \) (cf. Remark 2.6), we have \( A_0 + A_2 + B_2 = 5 \). From these three equations, we have \( A_2 + A_3 + B_1 = 6 \).

**Lemma 2.14.** \( \delta(C, D_4) \leq 19 \).

**Proof.** Let \( a_1 \) be the number of tangent lines that have contacts of order 3 at a point and intersect \( C \) transversally at two other points, and let \( a_i \) \((i = 2, 3)\) be the number of tangent lines that have contacts of order \( i \) at two points. Let \( b_1 \) be the number of bitangent lines that have contacts at two points, both of order 2, and let \( b_2 \) be the number of bitangent lines that have contacts at two points of orders 2 and 3. Let \( C^* \) be the dual curve of \( C \). Then, the degree of \( C^* \) is \( d^* = d(d-1) = 20 \), and its geometric genus equals \( (d-1)(d-2)/2 = 6 \) as the dual map \( C \to C^* \) is birational. Moreover, the number of double points of \( C^* \) (resp. triple points, singular points with multiplicity 4) equals \( a_1 + b_1 \) (resp. \( a_2 + b_2, a_3 \)), and every singular point of \( C^* \) can be resolved by one blowing up (cf. [6, Section 1.5]). By the genus formula, we have \( a_1 + b_1 + 3(a_2 + b_2) + 6a_3 = 165 \), and by the flex formula ([6, Example 1.5.11]), we have \( a_1 + b_2 + 2a_2 + 3a_3 = 45 \).

By Remark 2.11 and Lemma 2.13, we have \( 6\delta(C, D_4) \leq a_2 + a_3 + b_1 \). Therefore, \( 6\delta(C, D_4) \leq a_2 + a_3 + b_1 \leq a_2 + 3a_3 + b_1 + 2b_2 = (a_1 + b_1 + 3(a_2 + b_2) + 6a_3) - (a_1 + b_2 + 2a_2 + 3a_3) = 120 \), so that \( \delta(C, D_4) \leq 20 \).

We remark that if \( P \) and \( Q \) are distinct \( D_4 \)-points, then \( \iota_P(P) = P \) and \( \iota_P(Q) \neq Q \) by Lemma 2.8. Thus \( \delta(C, D_4) \) must be odd. Hence, we have \( \delta(C, D_4) \leq 19 \).

**3. Examples**

This section concerns the existence of plane quintic curves satisfying \( \delta(C, D_4) \in \{0, 1, 3, 5, 15\} \). In particular, we show that for any value of \( \delta(C, D_4) \) listed above, there exists a nonsingular plane quintic curve having the prescribed number of \( D_4 \)-points.
We have the classical theorem: the general smooth plane curve of degree \( d > 3 \) has no non-trivial automorphism. Hence, by Lemma 2.5, the general plane quintic curve does not possess \( D_4 \)-points, that is \( \delta(C, D_4) = 0 \).

On the other hand, Miura and Yoshihara proved the following result on \( D_4 \)-points on the Fermat quintic curve \( F(5) : X^5 + Y^5 + Z^5 = 0 \) (see [4, Theorem 1]).

**Theorem 3.1** ([4]). If \( P \in F(5) \) is not a flex of \( F(5) \), then the Galois group \( \text{Gal}(L_P/K_P) \) is isomorphic to the symmetric group of 4 letters and \( g(P) = 85 \). On the contrary, if \( P \in F(5) \) is a flex, then \( P \) is a \( D_4 \)-point and \( g(P) = 16 \).

In particular, the curve \( F(5) \) has 15 flexes, which lie on the three lines \( X = 0 \), \( Y = 0 \) and \( Z = 0 \). For example, it is easy to check that the point \( P = (0 : -1 : 1) \) is a \( D_4 \)-point for \( F(5) \). Indeed, by taking the projective transformation \( T : (X : Y : Z) \mapsto (\hat{X} : \hat{Y} : \hat{Z}) \) given by \( X = \hat{X}, Y = \hat{Y} + \hat{Z}, Z = \hat{Y} - \hat{Z} \), we have that \( T((0 : 0 : 1)) = P \) and \( T'(F(5)) : 10\hat{Y}^2\hat{Z}^4 + 20\hat{Y}^3\hat{Z}^2 + \hat{X}^5 + 2\hat{Y}^5 = 0 \). Thus \( F(5) \) satisfies \( \delta(C, D_4) = 15 \), and there are three disjoint 5-tuples of collinear \( D_4 \)-points.

Finally, we present three explicit examples of nonsingular plane quintic curves satisfying \( \delta(C, D_4) = 1, 3 \) and 5, respectively.

**Example 3.2.** Let \( C \) be the plane curve defined by the equation \( X^5 + Y^5 + Y^4Z + YZ^4 + Z^5 = 0 \) and \( P := (0 : -1 : 1) \). Then, \( C \) is a nonsingular plane quintic curve, \( \delta(C, D_4) = 1 \) and \( \Delta(C, D_4) = \{ P \} \).

We can prove that \( \delta(C, D_4) = 1 \) as follows. First, we can show easily that the point \( P \) is a \( D_4 \)-point. By taking the projective transformation \( T \) given by \( X = \hat{X}, Y = \hat{Y} + \hat{Z}, Z = \hat{Y} - \hat{Z} \), we have that \( T((0 : 0 : 1)) = P \) and \( T'(C) : (4\hat{Y}\hat{Z}^4 + (24\hat{Y}^3)\hat{Z}^2 + (\hat{X}^5 + 4\hat{Y}^5) = 0. \) Hence, by Theorem 2.1, \( P \) is a \( D_4 \)-point. Moreover, we have the involution \( \iota_P \) as follows:

\[
\iota_P = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.
\]

From a comparison of the Hessians of the defining equations, we infer that \( C \) is not projectively equivalent to \( F(5) \). By [8, Theorem 4.1 and Proposition 5.1], we have that the point \( Q := (1 : 0 : 0) \) is the unique outer Galois point for \( C \). Thus any \( \sigma \in \text{Aut}(C) \) satisfies \( \sigma(Q) = Q \). Then, by systematic direct computations, we can see that \( \iota_P \) is the only non-trivial involution of \( C \) fixing \( Q \). By Lemma 2.5, we conclude that \( \delta(C, D_4) = 1 \).

**Example 3.3.** Let \( C \) be the plane curve defined by the equation \( X^5 + Y^4Z + YZ^4 = 0 \) and \( P_i := (0 : -\omega^i : 1) \), where \( i = 1, 2, 3 \) and \( \omega \) is a primitive cubic root of unity. Then, \( C \) is a nonsingular plane quintic curve, \( \delta(C, D_4) = 3 \) and \( \Delta(C, D_4) = \{ P_1, P_2, P_3 \} \).

We can prove that \( \delta(C, D_4) = 3 \) as follows. First, we can show easily that 3 points \( P_1, P_2, P_3 \) are \( D_4 \)-points. For example, by taking the projective transformation \( T \) given by \( X = \hat{X}, Y = \hat{Y} + \hat{Z}, Z = \hat{Y} - \hat{Z} \), we have that \( T((0 : 0 : 1)) = P_1 \).
and $T^*(C) : (-6\tilde{Y})\tilde{Z}^4 + (4\tilde{Y}^3)\tilde{Z}^2 + (\tilde{X}^5 + 2\tilde{Y}^5) = 0$. Hence, by Theorem 2.1, $P_1$ is a $D_4$-point. Moreover, we have that the involutions $\iota_{P_1}$ and $\iota_{P_2}$ as follows:

$$
\iota_{P_1} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}, \quad \iota_{P_2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & \omega \\
0 & \omega^2 & 0
\end{pmatrix}.
$$

Hence, the order of $\iota_{P_1}|_{\ell} \circ \iota_{P_2}|_{\ell}$ equals 3, where $\ell : X = 0$. From a comparison of the Hessians of the defining equations, we infer that $C$ is not projectively equivalent to $F(5)$. By Lemma 2.10 and Theorem 1.2, we see that $\delta(C, D_4) = 3$.

**Example 3.4.** Let $C$ be the plane curve defined by the equation $X^5 + X^3YZ + Y^5 + Z^5 = 0$ and $P_1 := (0 : -\zeta^{i-1} : 1)$, where $i = 1, \ldots, 5$ and $\zeta$ is a primitive fifth root of unity. Then, $C$ is a nonsingular plane quintic curve, $\delta(C, D_4) = 5$ and $\Delta(C, D_4) = \{P_1, \ldots, P_5\}$.

We can prove that $\delta(C, D_4) = 5$ as follows. First, we can show easily that 5 points $P_1, \ldots, P_5$ are $D_4$-points. For example, by the projective transformation $T$ given by $X = \tilde{X}$, $Y = \tilde{Y} + \tilde{Z}$, $Z = \tilde{Y} - \tilde{Z}$, we have that $T((0 : 0 : 1)) = P_1$ and $T^*(C) : (10\tilde{Y})\tilde{Z}^4 + (-\tilde{X}^3 + 20\tilde{Y}^3)\tilde{Z}^2 + (\tilde{X}^5 + \tilde{X}^3\tilde{Y}^2 + 2\tilde{Y}^5) = 0$. Hence, by Theorem 2.1, $P_1$ is a $D_4$-point. Moreover, since $P_1$ is an ordinary flex and $F(5)$ has no ordinary flex, $C$ is not projectively equivalent to $F(5)$. By Theorem 1.2, we see that $\delta(C, D_4) = 5$.

**4. Proof of Theorem 1.2**

In this section, we prove Theorem 1.2. In particular, the proof follows almost straightforwardly from Lemmas 2.8 and 2.9, and from the following results.

**Lemma 4.1.** Assume that there exist five collinear $D_4$-points for $C$, as well as one other $D_4$-point. Then, $C$ is projectively equivalent to $F(5)$. In particular, $\delta(C, D_4) = 15$.

**Lemma 4.2.** Assume that $\#(\Delta(C, D_4) \cap \ell) \leq 3$ for every line $\ell$. Then, $\delta(C, D_4) \leq 3$.

In Subsections 4.1 and 4.2, we prove Lemmas 4.1 and 4.2, respectively. Subsection 4.3 is devoted to conclude the proof of Theorem 1.2.

**4.1 Proof of Lemma 4.1**

Assume that there exist five collinear $D_4$-points $P_i (i = 1, \ldots, 5)$ for $C$, as well as another $D_4$-point $P_0$. Taking a suitable projective transformation, we may assume that the five collinear points $P_i (i = 1, \ldots, 5)$ are on the line $\ell : X = 0$. Let $\sigma := \iota_{P_1} \circ \iota_{P_2} \in \text{Aut}(C)$. We will find other $D_4$-points using the automorphisms $\iota_{P_i} (i = 0, \ldots, 5)$ and $\sigma$. 
The three lines $\ell_{P_1}, \ell_{P_2}, \ell$ are not concurrent, i.e., they do not meet at one point.

**Proof.** Assume that $\ell_{P_1} \cap \ell_{P_2} \cap \ell = \{Q\}$. Taking a suitable projective transformation, we may assume that $P_1 = (0 : 0 : 1)$, $P_2 = (0 : 1 : 0)$, and $Q = (0 : 1 : 1)$. Then, as $\iota_{P_1}(P_1) = P_1$, $\iota_{P_1}(Q) = Q$, $\iota_{P_2}^2 = \text{id}$, $\iota_{P_2}(P_2) = P_2$, $\iota_{P_2}(Q) = Q$, and $\iota_{P_2}^2 = \text{id}$, we have that

$$\iota_{P_1}|_{\ell} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad \iota_{P_2}|_{\ell} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}. $$

Hence, the order of $\iota_{P_1}|_{\ell} \circ \iota_{P_2}|_{\ell}$ is not finite, which contradicts Lemma 2.10.

Let $Q_1$ be the intersection of $\ell_{P_1}$ and $\ell_{P_2}$. Then, $Q_1 \not\in \ell$ and $Q_1 \in \text{Fix}(\sigma)$. Consider the morphism $\nu : \ell \to \ell/\langle \sigma \rangle$, where the genera of $\ell$ and $\ell/\langle \sigma \rangle$ are $\deg(\ell) = g(\ell/\langle \sigma \rangle) = 0$, and $\deg \nu = 5$ by Lemma 2.10. Then Riemann-Hurwitz formula $2g(\ell) - 2 = (\deg \nu)(2g(\ell/\langle \sigma \rangle)) - 2) + \deg R$ assures that the ramification divisor $R$ consists of at least two points $Q_2, Q_3 \in \text{Fix}(\sigma) \cap \ell$. Taking a suitable projective transformation, we may assume that $Q_1 = (1 : 0 : 0)$, $Q_2 = (0 : 1 : 0)$, $Q_3 = (0 : 0 : 1)$. Then, $\sigma \in \text{PGL}(3, \mathbb{C})$ is expressed as a diagonal matrix.

Let $G := \langle \sigma \rangle$, $H := \langle \sigma^5 \rangle$, and $G|_{\ell} := \langle \sigma|_{\ell} \rangle$. Then, by Lemma 2.10, the order of $G|_{\ell}$ equals 5. Hence, we have the exact sequence $1 \to H \to G \to G|_{\ell} \to 1$ and $\#G|_{\ell} = 5$.

**Claim 4.4.** $\#G = 5$.

**Proof.** We show $\sigma^5 = \text{id}_C$. Because $Q_1 = (1 : 0 : 0) \in \ell_{P_1} \cap \ell_{P_2}$, $\iota_{P_1}(\ell) = \ell$, and $\iota_{P_2}(\ell) = \ell$, we have that the involutions $\iota_{P_1}$ and $\iota_{P_2}$ can be expressed as matrices:

$$\iota_{P_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & A_1 \end{pmatrix}, \quad \iota_{P_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & A_2 \end{pmatrix},$$

where $A_1, A_2 \in \text{GL}(2, \mathbb{C})$. We remark that $\iota_{P_i} (i = 1, 2)$ has a 1-dimensional eigenspace corresponding to $P_i$ and a 2-dimensional eigenspace corresponding to $\ell_{P_i}$ by Remark 2.3 and Lemma 2.5. The eigenvalues of $\iota_{P_i} (i = 1, 2)$ are equal to 1 and $-1$, since

$$\iota_{P_i}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & A_i^2 \end{pmatrix},$$

which induces $\text{id}_{P_2}$, must be a unit matrix. The eigenvalue of $\iota_{P_i} (i = 1, 2)$ belonging to $Q_1 = (1 : 0 : 0) \in \ell_{P_i}$ equals 1. Thus, we infer that the eigenvalues of $A_i (i = 1, 2)$ are equal to 1 and $-1$. In particular, $\det A_i = -1 (i = 1, 2)$.

Because

$$\sigma^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & (A_1 A_2)^5 \end{pmatrix}, \quad \sigma^5|_{\ell} = \text{id}_\ell \quad \text{and} \quad \det(A_1 A_2)^5 = 1,$$
we have that
\[
\sigma^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

Assume that \(\sigma^5 \neq \text{id}_C\). Then, by Lemma 2.7, the defining equation of \(C\) can be expressed as \(f_1(Y, Z)X^4 + f_3(Y, Z)X^2 + f_5(Y, Z) = 0\). In particular, we have that \(f_5(P_1) = 0\) as \(P_1 \in C \cap \{X = 0\}\), and the line \(\ell(P_1, Q_1)\) passing through \(Q_1\) and \(P_1\) is given by a factor of \(f_5\). From \(I_{P_1}(C, \ell(P_1, Q_1)) \geq 2\), we see that \(\ell(P_1, Q_1) = T_{P_1}C, \) and \(I_{P_1}(C, T_{P_1}C) = 2\) or 4. This contradicts Lemma 2.4.

Since \(\sigma\) is represented by a diagonal matrix in \(PGL(3, \mathbb{C})\) and \(\sigma \neq \text{id}_{p_2}\), Remark 2.3 assures that its fixed locus \(\text{Fix}(\sigma)\) consists of either three non-collinear points, or of one point and one line. In the latter case we have:

Claim 4.5. If \(\text{Fix}(\sigma)\) consists of one point and one line, then \(Q_2\) and \(Q_3\) are Galois points.

Proof. Because \(Q_1, Q_2, Q_3 \in \text{Fix}(\sigma)\), we may assume that \(\text{Fix}(\sigma) = \{Q_2\} \cup \ell(Q_1, Q_3)\), where \(\ell(Q_1, Q_3)\) is the line passing through \(Q_1\) and \(Q_3\). By Claim 4.4, the morphism \(C \to C/\langle \sigma \rangle\) is a cyclic covering of degree 5, and its ramification divisor consists of 5 total ramification points \(\ell(Q_1, Q_3) \cap C\). From the Riemann-Hurwitz formula, we see that the quotient \(C/\langle \sigma \rangle\) is isomorphic to \(\mathbb{P}^1\). Hence, by [5, Proposition 2.3.6], the covering \(C \to C/\langle \sigma \rangle\) is obtained as a projection. Because \(\sigma(\ell') = \ell'\) for every line \(\ell'\) passing through \(Q_2\), we see that the center of the projection \(C \to C/\langle \sigma \rangle\) is the point \(Q_2\). Therefore, \(Q_2\) is a Galois point.

From \(\sigma \circ \iota_{P_2}(Q_2) = \iota_{P_2}(Q_2) = \iota_{P_2} \circ \sigma(Q_2) = \iota_{P_2}(Q_2)\) and \(\iota_{P_1} \cap \iota_{P_2} \cap \ell = \emptyset\), we infer that \(\iota_{P_2}(Q_2) = Q_3\). Hence, the point \(Q_3\) is also a Galois point.

In particular, \(Q_2, Q_3 \in \mathbb{P}^2 \setminus C\). Therefore, [8, Theorem 4' and Proposition 5'] assure that if \(\text{Fix}(\sigma)\) consists of one point and one line, then \(C\) is projectively equivalent to \(F(5)\). Hence we only need to prove the assertion of Lemma 4.1 when \(\text{Fix}(\sigma) = \{Q_1, Q_2, Q_3\}\) consists of three non-collinear points.

Claim 4.6. For a point \(P \in \mathbb{P}^2 \setminus \{Q_1, Q_2, Q_3\}\), the five points \(P, \sigma(P), \sigma^2(P), \sigma^3(P), \sigma^4(P)\) are collinear if and only if \(P \in V(XYZ)\). Furthermore, if \(P \notin V(XYZ)\), then no three of these five points can be collinear.

Proof. Assume that \(P \in V(XYZ)\). Because \(\text{Fix}(\sigma) = \{Q_1, Q_2, Q_3\}\), we have \(\sigma(V(X)) = V(X), \sigma(V(Y)) = V(Y), \) and \(\sigma(V(Z)) = V(Z)\). Hence, \(P, \sigma(P), \sigma^2(P), \sigma^3(P), \sigma^4(P)\) are collinear.

Assume that \(P \notin V(XYZ)\). Let \(P = (1 : a : b)\), where \(ab \neq 0\). As \(\sigma^5 = \text{id}_C\) by Claim 4.4, \(\sigma\) can be expressed as \((X : Y : Z) \mapsto (X : \zeta Y : \zeta^2 Z)\), where \(i = 2, 3,\) or 4. Then, the three points \(P = (1 : a : b)\), \(\sigma(P) = (1 : \zeta a : \zeta^2 b)\), \(\sigma^2(P) = (1 : \zeta^2 a : \zeta^3 b)\) cannot be collinear.

Claim 4.7. There exist 15 \(D_4\)-points \(P_1, \ldots, P_{15}\) such that each five points \(P_{i+1}, \ldots, P_{i+5}\) \((i = 0, 5, 10)\) are collinear.
The five $D_4$-points $P_1, \ldots, P_5$ are collinear by assumption. Using the involution induced by $D_4$-point $P_0$ not collinear to them, we have that $\iota_{P_0}(P_i)$, $i = 1, \ldots, 5$, are collinear $D_4$-points. Let $P_{5+i} := \iota_{P_0}(P_i)$ ($i = 1, \ldots, 5$). Then, $\iota_{P_0}(P_1), \ldots, \iota_{P_0}(P_5)$ are also collinear $D_4$-points. Let $P_{10+i} := \iota_{P_0}(P_i)$ ($i = 1, \ldots, 5$). Note that $P_1, \ldots, P_{15}$ are distinct by Lemma 2.8 (and the point $P_{11}$ is actually $P_0$).

Claim 4.8. If $\{P_6, \ldots, P_{15}\} \not\subset V(YZ)$, then $\delta(C, D_4) \geq 30$.

**Proof.** We may assume that $\{P_6, \ldots, P_{10}\} \not\subset V(YZ)$. Let $S_i := \{P_i, \sigma(P_i), \ldots, \sigma^4(P_i)\}$ ($i = 6, \ldots, 10$).

We show $S_i \cap S_j = \emptyset$ for $i \neq j$. Assume that $S_i \cap S_j \neq \emptyset$. Then, there exists a point $P \in S_i \cap S_j$, and then $\{P, \sigma(P), \ldots, \sigma^4(P)\} = S_i = S_j$. As $\iota_{P_0}(P_6) = P_0$, we have that $\iota_{P_0}(S_6) = S_6$. Hence, $\iota_{P_0}(S_7) = S_6 \uplus P_0, P_7$, where the three points $P_6, P_7, \iota_{P_0}(P_7)$ are collinear. By Claim 4.6, $P_6, P_7 \in V(YZ)$. Hence, $S_6 = C \cap V(Y)$ or $S_6 = C \cap V(Z)$. Because $P_6, P_7 \in S_6$ and $P_6, \ldots, P_{10}$ are collinear, we have that $P_6, \ldots, P_{10} \in V(YZ)$, which is a contradiction. By the same argument as above, we see that $S_i \cap S_j = \emptyset$ for $i \neq j$.

The number of points in $\{P_1, \ldots, P_5\} \cup S_6 \cup \cdots \cup S_{10}$ equals 30.

By Lemma 2.14, Claim 4.7, and Claim 4.8, we have that the $D_4$-points $P_1, \ldots, P_{15}$ are on $V(XYZ)$.

Claim 4.9. $\delta(C, D_4) = 15$.

**Proof.** If $\delta(C, D_4) > 15$, then we have 15 $D_4$-points $P_1, \ldots, P_{15}$ on $V(XYZ)$, and another $D_4$-point $P_{16} \not\in V(XYZ)$. Then, by Claim 4.4 and Fix($\sigma$) = $\{Q_1, Q_2, Q_3\}$, we have that $P_i$, where $i = 1, \ldots, 15$, and $\sigma^j(P_{16})$, where $j = 0, \ldots, 4$, are distinct $D_4$-points. This contradicts Lemma 2.14.

Claim 4.10. Taking a suitable projective transformation, we may assume that $\{P_1, \ldots, P_{15}\} = F(5) \cap V(XYZ)$.

**Proof.** We may assume that $\{P_1, \ldots, P_3\} \subset V(X)$, $\{P_6, \ldots, P_{10}\} \subset V(Y)$ and $\{P_1, \ldots, P_{15}\} \subset V(Z)$. Moreover, we may assume that $P_1 = (0 : -1 : 1)$ and $P_6 = (-1 : 0 : 1)$. Note that $\iota_{P_i}(\Delta(C, D_4)) = \Delta(C, D_4)$ for any $i = 1, \ldots, 15$ by Claim 4.9, and $Q_1 = (1 : 0 : 0), Q_2 = (0 : 1 : 0), Q_3 = (0 : 0 : 1)$. Thus $\iota_{P_i}(\{Q_1, Q_2, Q_3\}) = \{Q_1, Q_2, Q_3\}$ for any $i = 1, \ldots, 15$. Because $\iota_{P_i}(P_j) \neq P_j$ for $i \neq j$ ($i, j \in \{1, \ldots, 15\}$), and $\iota_{P_i}(\{P_1, \ldots, P_3\}) = \{P_1, \ldots, P_3\}$ for any $i \in \{1, \ldots, 5\}$, we have that $\iota_{P_i}(Q_1) = Q_1, \iota_{P_i}(Q_2) = Q_2$ for any $i \in \{1, \ldots, 5\}$.

Because $\iota_{P_i}(Q_1) = Q_1, \iota_{P_i}(Q_2) = Q_2, \iota_{P_i}(P_1) = P_1$ and $\iota_{P_i}^2 = id_C$, we have that

\[
\iota_{P_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.
\]
By Lemma 2.5, the eigenspace corresponding \( P_1 \) is 1-dimensional. Thus we have that
\[
\iota_{P_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

Hence, we may assume that \( P_2 = (0 : a : 1), P_3 = (0 : 1 : a), P_4 = (0 : b : 1), \)
\( P_5 = (0 : 1 : b) \), where \( a, b \in \mathbb{C} \setminus \{0\} \) and \( a \neq b \). Moreover, we may assume that \( \iota_{P_2}(P_1) = P_4 \) and \( \iota_{P_5}(P_1) = P_5 \). Then, as \( \iota_{P_2}(Q_1) = Q_1, \iota_{P_2}(Q_2) = Q_3, \)
\( \iota_{P_2}(P_2) = P_2, \iota_{P_5}(P_1) = P_4, \) and by Lemma 2.5 (i.e., the eigenspace corresponding to \( P_2 \) is 1-dimensional), we have that
\[
\iota_{P_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -a \\ 0 & -1/a & 0 \end{pmatrix}
\]
and \( b = -a^2 \). Because \( \iota_{P_2}(P_3) = P_5 \), we have that \( a^5 = -1 \). Hence, we have that
\( P_1 = (0 : -1 : 1), P_2 = (0 : -\zeta : 1), P_3 = (0 : -\zeta^4 : 1), P_4 = (0 : -\zeta^2 : 1), \)
\( P_5 = (0 : -\zeta^3 : 1), \) where \( \zeta \) is a primitive fifth root of unity.

By the same argument as above, we have that \( P_6 = (-1 : 0 : 1), P_7 = (-\zeta : 0 : 1), P_8 = (-\zeta^4 : 0 : 1), P_9 = (-\zeta^2 : 0 : 1), P_{10} = (-\zeta^3 : 0 : 1). \) By
\( \iota_{P_6}(V(Y) \cap C) = V(Z) \cap C, \) we have that \( P_{11} = (-1 : 1 : 0), P_{12} = (-\zeta : 1 : 0), \)
\( P_{13} = (-\zeta^4 : 1 : 0), P_{14} = (-\zeta^2 : 1 : 0), P_{15} = (-\zeta^3 : 1 : 0). \) Note that
\( F(5) \cap V(XYZ) \) is the set of these 15 points.

By a similar argument to that in the proof of Claim 4.10, we have that
\[
\iota_{P_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \iota_{P_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\zeta \\ 0 & -\zeta^4 & 0 \end{pmatrix}, \iota_{P_6} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

Let the defining equation of \( C \) be expressed by \( F(X, Y, Z) = \sum_{i=0}^{5} X^{5-i} f_i(Y, Z). \)
Because \( Q_1 = (1 : 0 : 0) \notin C, \) we may assume that \( f_0(Y, Z) = 1. \) As \( \iota_{P_6}(C) = C \)
\((i = 1, 2), \) we have that \( \iota_{P_6}^* F = \lambda_{i} F, \) where \( \lambda_i \in \mathbb{C} \setminus \{0\}. \) Using the matrices above as representations of \( \iota_{P_i} (i = 1, 2, 6), \) we have that \( \lambda_1 = \lambda_{2} = 1. \) Hence, each \( f_i \) \((i = 1, \ldots, 5) \) is symmetric and \( \iota_{P_i}^* f_i = f_i. \) Therefore, \( F = X^5 + \alpha X^3 Y Z + \beta X Y^2 Z^2 + \gamma Y^5 + \gamma Z^5, \) where \( \alpha, \beta, \gamma \in \mathbb{C}. \) As \( P_5 = (-1 : 0 : 1) \in C, \) we have
\( \gamma = 1, \) and because \( \iota_{P_5}^* F = \lambda_{0} F \) for some \( \lambda_0 \in \mathbb{C} \setminus \{0\}. \) we have that \( \alpha = \beta = 0. \)
Namely, \( C \) is the Fermat quintic curve. This completes the proof of Lemma 4.1.

4.2 - Proof of Lemma 4.2

In order to prove Lemma 4.2, we assume that \( \#(\Delta(C, D_4) \cap \ell) \leq 3 \) for every line \( \ell \subset \mathbb{P}^2. \) Aiming for a contradiction, we assume further that \( \delta(C, D_4) > 3. \)

Claim 4.11. There exists a finite subset \( S \subset \Delta(C, D_4) \) such that \# \( S = 9 \) and
\( \#(S \cap \ell) = 0, 1, \) or 3 for every line \( \ell. \)
Projection of a nonsingular plane quintic curve

Let \( L_{ijk} \) be the line passing through the three collinear points \( P_1, P_2, P_3 \).

We can take two \( D_1 \)-points \( P_1 \) and \( P_2 \). By Lemma 2.8, we can find a third point \( P_3 := \iota_{P_1}(P_2) \) such that \( P_1, P_2, P_3 \) are collinear. By our assumption, \( \Delta(C, D_4) \cap L_{123} = \{ P_1, P_2, P_3 \} \). However, \( \delta(C, D_4) > 3 \), there exists a point \( P_0 \in \Delta(C, D_4) \setminus L_{123} \). Let \( P_{i+3} := \iota_{P_0}(P_i) \) (\( i = 1, 2, 3 \)). Then, the three points \( \{ P_0, P_3, P_6 \} \) (resp. \( \{ P_0, P_1, P_4 \}, \{ P_0, P_2, P_5 \}, \{ P_0, P_3, P_6 \}) \) are collinear \( D_1 \)-points, and \( \Delta(C, D_4) \cap L_{456} = \{ P_0, P_3, P_6 \} \) (resp. \( \Delta(C, D_4) \cap L_{014} = \{ P_0, P_1, P_4 \}, \Delta(C, D_4) \cap L_{025} = \{ P_0, P_2, P_5 \}, \Delta(C, D_4) \cap L_{036} = \{ P_0, P_3, P_6 \}) \). Note that \( P_0, \ldots, P_6 \) are seven distinct points. Let \( P_7 := \iota_{P_0}(P_1) \) and \( P_8 := \iota_{P_0}(P_3) \). Then, as \( L_{157} \) (resp. \( L_{358} \)) \( \neq L_{123}, L_{456}, L_{014}, L_{025}, L_{036} \), we see that \( P_7 \) (resp. \( P_8 \)) \( \neq P_0, \ldots, P_6 \). Because \( P_5 \notin \{ P_0, \ldots, P_6 \} \), we see that \( P_7 \neq P_8 \). Hence, \( P_0, \ldots, P_8 \) are nine distinct points. Let \( S := \{ P_0, \ldots, P_8 \} \).

We show that each set of three points \( \{ P_0, P_1, P_8 \}, \{ P_2, P_4, P_8 \}, \{ P_2, P_5, P_7 \}, \{ P_1, P_3, P_8 \}, \{ P_3, P_4, P_7 \}, \{ P_1, P_6, P_7 \} \) are collinear.

Consider the three points \( \{ P_0, P_7, P_3 \} \). Because \( \Delta(C, D_4) \cap L_{025} = \{ P_0, P_2, P_3 \} \), we have that \( \iota_{P_0}(P_2) = P_0 \). Hence, \( P_0, P_7, P_3 \in \iota_{P_0}(L_{123}) \), i.e., \( \{ P_0, P_7, P_3 \} \) are collinear, and \( \Delta(C, D_4) \cap L_{078} = \{ P_0, P_7, P_3 \} \).

Next, consider the three points \( \{ P_2, P_4, P_8 \} \). As \( \Delta(C, D_4) \cap L_{014} = \{ P_0, P_1, P_4 \} \) (resp. \( \Delta(C, D_4) \cap L_{157} = \{ P_1, P_0, P_4 \}) \), we have that \( \iota_{P_0}(P_1) = P_0 \) (resp. \( \iota_{P_0}(P_3) = P_0 \)). Hence, \( \iota_{P_0}(P_2) \in \iota_{P_0}(L_{456}) = L_{078} \), and so \( \iota_{P_0}(P_2) \in L_{078} \). Moreover, as \( \iota_{P_0}(P_2) = P_2 \) and \( \iota_{P_3}(P_0) = P_1 \), we have that \( \{ P_2, P_4, P_8 \} \subset \iota_{P_0}(L_{036}) \). Hence, \( \{ P_2, P_4, P_8 \} \) are collinear, and \( \Delta(C, D_4) \cap L_{248} = \{ P_2, P_4, P_8 \} \).

By the same argument as above, we see that \( \{ P_2, P_6, P_7 \} = \iota_{P_0}(\{ P_0, P_1, P_4 \} \subset \iota_{P_0}(L_{014}) \), and \( \{ P_2, P_4, P_6 \} = \iota_{P_0}(\{ P_0, P_2, P_5 \} \subset \iota_{P_0}(L_{025}) \), and \( \{ P_2, P_6, P_7 \} = \iota_{P_0}(\{ P_0, P_3, P_6 \} \subset \iota_{P_0}(L_{036}) \), and \( \{ P_2, P_4, P_6 \} = \iota_{P_0}(\{ P_0, P_4, P_7 \} \subset \iota_{P_0}(L_{078}) \), and \( \{ P_2, P_4, P_6 \} \) are collinear.

Therefore, \( \#(S \cap \ell) = 0, 1, 3 \) for every line \( \ell \).

Let \( P_0, P_3, P_6 \) and \( L_{ijk} \) be those in the proof of Claim 4.11.

Claim 4.12. The three lines \( L_{123}, L_{456}, L_{078} \) are not concurrent, i.e., they do not meet at one point.

Proof. Assume that \( L_{123} \cap L_{456} \cap L_{078} = \{ Q \} \). Because \( \iota_{P_0}(L_{123}) = L_{456} \) and \( \iota_{P_0}(L_{456}) = L_{123} \) (\( i = 0, 7 \)), we have that \( \iota_{P_0}(Q) = Q \) and \( \iota_{P_0}(Q) = Q \). Let \( C \cap L_{078} := \{ P_0, P_7, P_8, Q_1, Q_2 \} \). Then, as \( \iota_{P_0}(Q_1) = Q_2 \), we have that \( Q \neq Q_1, Q_2 \). Note that \( \iota_{P_3}(L_{078}) \) and \( \iota_{P_0}(L_{078}) = L_{078} \). Let \( \sigma := (\iota_{P_0} \circ \iota_{P_3})|_{L_{078}} \in PGL(2, C) \). Since \( \iota_{P_0}(Q_1) = \iota_{P_3}(Q_1) = Q_2 \), we have \( Q, Q_1, Q_2 \in \text{Fix}(\sigma) := \{ R \in L_{078} | \sigma(R) = R \} \). Hence, \( \sigma = \text{id}_{L_{078}} \). However, we have that \( \sigma(P_3) = P_8 \), which is a contradiction.

Claim 4.13. Taking a suitable projective transformation, we may assume the following:

\[
\begin{align*}
P_0 &= (-1 : 1 : 0), \quad P_7 = (-\omega^2 : 1 : 0), \quad P_8 = (-\omega : 1 : 0), \\
P_1 &= (-1 : 0 : 1), \quad P_2 = (-\omega : 0 : 1), \quad P_3 = (-\omega^2 : 0 : 1), \\
P_4 &= (0 : -1 : 1), \quad P_5 = (0 : -\omega : 1), \quad P_6 = (0 : -\omega^2 : 1),
\end{align*}
\]
where \( \omega \) is a primitive cubic root of unity.

**Proof.** Let \( \{Q_1\} := L_{123} \cap L_{456}, \{Q_2\} := L_{456} \cap L_{078}, \) and \( \{Q_3\} := L_{078} \cap L_{123}. \) By Claim 4.12, \( Q_1, Q_2, Q_3 \) are not collinear, distinct points. Taking a suitable projective transformation, we may assume that

\[
Q_1 = (0 : 0 : 1), \quad Q_2 = (0 : 1 : 0), \quad Q_3 = (1 : 0 : 0).
\]

Then,

\[
L_{456} = V(X), \quad L_{123} = V(Y), \quad L_{078} = V(Z).
\]

Note that \( Q_i \notin \{P_0, \ldots, P_8\} \ (i = 1, 2, 3) \). Let \( P_i = (0 : a_i : 1) \ (i = 4, 5, 6), \) where \( a_i \in \mathbb{C} \setminus \{0\}. \) Taking the projective transformation \( (X : Y : Z) \mapsto (X : -1/a_4 Y : Z) \), we may assume that \( a_4 = -1 \). From \( \iota_{P_4}(L_{456}) = L_{456}, \ i\iota_{P_4}(L_{123}) = L_{078}, \ i\iota_{P_4} = \text{id}_C, \ i\iota_{P_4}(P_4) = P_4, \) and \( P_4 \notin \overline{\ell}_{P_4}, \) we infer that

\[
\iota_{P_4} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.
\]

Hence, \( P_6 = (0 : a_6 : 1) = \iota_{P_4}(P_5) = (0 : 1 : a_5), \) and so \( a_5a_6 = 1. \)

By a similar argument to that above (note that \( \iota_{P_5}(P_4) = P_5 \)), we infer that

\[
\iota_{P_5} = \begin{pmatrix}
\sqrt{-a_6} & 0 & 0 \\
0 & 0 & -a_6 \\
0 & 1 & 0
\end{pmatrix}.
\]

Hence, \( P_5 = (0 : a_5 : 1) = (0 : -a_6 : a_5), \) and so \( a_5^2 = -a_6. \) We have that \( a_5^3 = -1. \) Note that \( P_5 \neq P_4, \) that is, \( a_5 \neq -1. \) Hence, we may put \( a_5 = -\omega \) and \( P_4 = (0 : -1 : 1), P_5 = (0 : -\omega : 1), P_6 = (0 : -\omega^2 : 1). \)

Again, using a similar argument, we may assume that \( P_1 = (-1 : 0 : 1), \ P_2 = (-\omega : 0 : 1), P_3 = (-\omega^2 : 0 : 1). \) As \( P_0 \in L_{078} \cap L_{014}, P_7 \in L_{078} \cap L_{267}, \) \( P_8 \in L_{078} \cap L_{358}, \) we have that \( P_0 = (-1 : 1 : 0), P_7 = (-\omega^2 : 1 : 0), P_8 = (-\omega : 1 : 0). \)

**Claim 4.14.** Assume that \( P_0, \ldots, P_8 \) are the points in Claim 4.13. Then,

\[
\iota_{P_0} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \iota_{P_1} = \begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & \omega & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \iota_{P_2} = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \iota_{P_3} = \begin{pmatrix} 0 & 0 & \omega \\ 0 & \omega & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \iota_{P_4} = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega \\ 0 & 1 & 0 \end{pmatrix}, \quad \iota_{P_5} = \begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & \omega & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]
From (4.1), we have that
\[ \text{Fix}(\mathbf{P}_0) = \mathbf{P}_0 \]
\[ \text{Fix}(\mathbf{P}_1) = \mathbf{P}_1 \]
\[ \text{Fix}(\mathbf{P}_2) = \mathbf{P}_2 \]
\[ \text{Fix}(\mathbf{P}_3) = \mathbf{P}_3 \]
\[ \text{Fix}(\mathbf{P}_4) = \mathbf{P}_4 \]
\[ \text{Fix}(\mathbf{P}_5) = \mathbf{P}_5 \]
\[ \text{Fix}(\mathbf{P}_6) = \mathbf{P}_6 \]
\[ \text{Fix}(\mathbf{P}_7) = \mathbf{P}_7 \]
\[ \text{Fix}(\mathbf{P}_8) = \mathbf{P}_8 \]

Then, we obtain the cyclic Galois covering \( \mathbf{P} \) given by (4.1).

\[ \text{Proof.} \]

However, by the Riemann–Hurwitz formula, this is a contradiction. Hence, we have that \( \text{Fix}(\mathbf{P}_0) = \mathbf{P}_0 \) and the same would hold for \( \mathbf{P}_1 \).

By a similar argument to that above, we can determine the other \( \mathbf{P}_i \).

\[ \text{Claim 4.15.} \] Assume that \( P_0, \ldots, P_8 \) are the points in Claim 4.13. Then, \( (0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0) \in C \).

\[ \text{Proof.} \] Note that \( \#(C \cap L_{078}) = 5 \) by Remark 2.11. Let \( C \cap L_{078} = \{P_0, P_7, P_8, Q_1, Q_2\} \). As neither \( Q_1 \) nor \( Q_2 \) are \( D_4 \)-points, we have that \( \text{Fix}(\mathbf{P}_i) = Q_2 \) \( (i = 0, 7, 8) \). Let \( \sigma := \text{Fix}(\mathbf{P}_i) \circ \text{Fix}(\mathbf{P}_i) \). Then, \( Q_1, Q_2 \in \text{Fix}(\sigma) \cap L_{078} \).

Because

\[ \sigma = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

we have that \( \text{Fix}(\sigma) \cap L_{078} = \{(1 : 0 : 0), (0 : 1 : 0)\} \). Hence, \( \{(1 : 0 : 0), (0 : 1 : 0)\} \subset C \). Thus \( \text{Fix}(\sigma)((0 : 1 : 0)) = (0 : 0 : 1) \in C \).

In order to complete the proof of Lemma 4.2, let us consider the points \( P_0, \ldots, P_8 \) listed in Claim 4.13, and let \( \sigma := \text{Fix}(\mathbf{P}_i) \circ \text{Fix}(\mathbf{P}_i) \in \text{Aut}(C) \) with representation matrix given by (4.1). Then, we obtain the cyclic Galois covering \( C \rightarrow C/\langle \sigma \rangle \) of degree 3, whose ramification points are the three points \( (0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0) \in C \). However, by the Riemann–Hurwitz formula, this is a contradiction. Hence, we have \( \delta(C, D_4) \leq 3 \), which concludes Lemma 4.2.

4.3 – Proof of Theorem 1.2

Let us prove \( \delta(C, D_4) \in \{0, 1, 3, 5, 15\} \). Clearly, \( \delta(C, D_4) \neq 2 \) by Lemma 2.8.

If \( \delta(C, D_4) \) were equal to 4, then \( \delta(C, D_4) \) would consist of 4 collinear points \( P_1, \ldots, P_4 \). Indeed, \( P_1, P_2 \) and \( P_3 := \text{Fix}(\mathbf{P}_3) \) would be collinear by Lemma 2.8, and the same would hold for \( P_1, P_2 \) and \( \text{Fix}(\mathbf{P}_1) \cap \text{Fix}(\mathbf{P}_2) \in \{P_1, P_2\} \). Thus we would get a contradiction as Lemma 2.9 would imply \( \delta(C, D_4) \geq 5 \).

Finally, if \( \delta(C, D_4) \geq 6 \), then Lemmas 2.9 and 4.2 give that there exists a line \( \ell \) passing exactly through five \( D_4 \)-points. So there exists another \( D_4 \)-point not lying on \( \ell \), and Lemma 4.1 implies that \( \delta(C, D_4) = 15 \) and \( C \) is projectively equivalent to the Fermat quintic curve \( F(5) \). Conversely, if \( C \) is projectively equivalent to the Fermat quintic curve \( F(5) \), by the analysis of Section 3 we deduce that \( \delta(C, D_4) = 15 \), and \( C \) possesses three disjoint 5-tuples of collinear \( D_4 \)-points.

In order to conclude the proof of Theorem 1.2, we only need to show that if \( \delta(C, D_4) = 3 \) or 5, then \( \Delta(C, D_4) \) lies on a line. The case \( \delta(C, D_4) = 3 \) follows straightforwardly from Lemma 2.8, whereas Lemmas 2.9 and 4.2 give the assertion when \( \delta(C, D_4) = 5 \).
References


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