Rend. Sem. Mat. Univ. Padova 1xx (201x) **Rendiconti del Seminario Matematico della Università di Padova**

Existence of stationary solutions for some integro-differential equations with superdiffusion

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Abstract – The work deals with the existence of solutions of an integro-differential equation arising in population dynamics in the case of anomalous diffusion. The proof of existence of solutions relies on a fixed point technique. Solvability conditions for non-Fredholm elliptic operators in unbounded domains are being used.

Mathematics Subject Classification (2010). 35J05; 35P30, 47F05.

Keywords. integro-differential equations, non Fredholm operators, Sobolev spaces

1. Introduction

In the present article we study the existence of stationary solutions of the integro-differential equation

$$\frac{\partial u}{\partial t} = -D\sqrt{-\Delta}u + \int_{\mathbb{R}^d} K(x-y)g(u(y,t))dy + f(x), \tag{1}$$

which appears in cell population dynamics. The space variable x is correspondent to the cell genotype, $u(x,t)$ stands for the cell density as a function of their genotype and time. The right side of this equation describes the evolution of cell density due to cell proliferation, mutations and cell influx. Namely, the anomalous

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diffusion term corresponds to the change of genotype via small random mutations, and the nonlocal production term describes large mutations. In this context $g(u)$ is the rate of cell birth which depends on u (density dependent proliferation), and the function $K(x-y)$ shows the proportion of newly born cells which change their genotype from y to x. We assume that it depends on the distance between the genotypes. Finally, the last term in the right-hand side of this equation describes the influx or efflux of cells for different genotypes.

The square root of Laplacian in equation (1) represents a particular case of superdiffusion intensively studied in relation with various applications in plasma physics and turbulence [13], [14], surface diffusion [15], [16], semiconductors [17] and so on. The physical meaning of superdiffusion is that the random process occurs with longer jumps in comparison with normal diffusion. The moments of jump length distribution is finite in the case of normal diffusion, but this is not the case for superdiffusion. The operator $\sqrt{-\Delta}$ is defined via the spectral calculus. A similar equation in the case with the standard Laplacian in the diffusion term was treated recently in [28].

Further we will set $D = 1$ and will explore the existence of solutions of the equation

$$-\sqrt{-\Delta}u + \int_{\mathbb{R}^d} K(x-y)g(u(y))dy + f(x) = 0, \tag{2}$$

which is viewed as a perturbation of problem (11). Let us consider the case in which the linear part of this operator fails to satisfy the Fredholm property, such that conventional methods of nonlinear analysis may not be applicable. We will use solvability conditions for non Fredholm operators along with the method of contraction mappings.

Consider the problem

$$-\Delta u + V(x)u - au = f, \tag{3}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ is either zero identically or converges to 0 at infinity. For $a \geq 0$, the essential spectrum of the operator $A : E \to F$ which corresponds to the left side of equation (3) contains the origin. Consequently, such operator fails to satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present work is devoted to the studies of some properties of the operators of this kind. Note that elliptic problems with non Fredholm operators were studied extensively in recent years. Approaches in weighted Sobolev and Hölder spaces were developed in [2], [3], [4], [5], [6]. The Schrödinger type operators without Fredholm property were treated via the methods of the spectral and the scattering theory in [18], [21]. The Laplacian operator with drift from the point of view of non Fredholm operators was studied in [22] and linearized Cahn-Hilliard equations in [24] and [26]. Nonlinear non Fredholm elliptic problems were treated in [25] and [27]. Important applications to the theory of reaction-diffusion problems were developed in [10], [11]. Non Fredholm operators arise also when studying wave systems with an infinite number of localized traveling waves (see [1]). In particular, when $a = 0$ the operator A is Fredholm in some properly chosen weighted spaces (see [2], [3], [4], [5], [6]). However, the case of $a \neq 0$ is considerably different and the approach developed in these works cannot be applied. In [8] the authors prove regularity results concerning the solutions of problems involving the fractional Laplacian by exploiting purely local techniques. The influence of fractional diffusion in Fisher-KPP equations was well understood in [7]. Front propagation problems with superdiffusion were studied extensively in recent years (see e.g. [29], [30]).

Let us set $K(x) = \varepsilon\mathcal{K}(x)$, where $\varepsilon \geq 0$, such that the reproduction rate is much less than that of diffusion and cell influx or efflux and suppose that the assumption below is fulfilled.

Assumption 1.1. *Let $f(x) : \mathbb{R}^3 \to \mathbb{R}$ be nontrivial, $f(x) \in L^1(\mathbb{R}^3)$ and $\nabla f(x) \in L^2(\mathbb{R}^3)$. Assume also that $\mathcal{K}(x) : \mathbb{R}^3 \to \mathbb{R}$ and $\mathcal{K}(x) \in L^1(\mathbb{R}^3)$.*

We choose the space dimension $d = 3$, which is related to the solvability conditions for the linear Poisson equation (11) discussed in Lemma 3.1. Our results obtained below can be generalized to $d > 3$ using similar ideas but somewhat different techniques, for instance in the sense of Sobolev embeddings, which are dimension dependent. From the perspective of applications, the space dimension is not limited to $d = 3$ due to the fact that the space variable corresponds to cell genotype but not to the usual physical space.

By means of the standard Sobolev inequality (see e.g. p.183 of [12]) under the assumption given above we obtain

$$f(x) \in L^2(\mathbb{R}^3).$$

We consider the Sobolev space

$$H^2(\mathbb{R}^3) := \{u(x) : \mathbb{R}^3 \to \mathbb{C} \mid u(x) \in L^2(\mathbb{R}^3),\ \Delta u \in L^2(\mathbb{R}^3)\}$$

equipped with the norm

$$\|u\|^2_{H^2(\mathbb{R}^3)} := \|u\|^2_{L^2(\mathbb{R}^3)} + \|\Delta u\|^2_{L^2(\mathbb{R}^3)}. \tag{4}$$

The Sobolev embedding yields

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq c_e\|u\|_{H^2(\mathbb{R}^3)}, \tag{5}$$

where $c_e > 0$ is the constant of the embedding. When the nonnegative parameter ε vanishes, we arrive at the linear Poisson equation (11). By means of Lemma 3.1 below under our Assumption 1.1 problem (11) admits a unique solution $u_0(x) \in H^1(\mathbb{R}^3)$ and no orthogonality relations are required. Lemma 3.1 yields that in dimensions $d < 3$ we need specific orthogonality conditions to be able to solve equation (11) in $H^1(\mathbb{R}^d)$. Let us not treat the problem in dimensions $d > 3$ to avoid additional technicalities due to the fact that the proof will rely on similar ideas (see Lemma 3.1). By virtue of Assumption 1.1, using that

$$\|\Delta u_0\|^2_{L^2(\mathbb{R}^3)} = \|\nabla f(x)\|^2_{L^2(\mathbb{R}^3)}, \tag{6}$$

we obtain for the unique solution of our linear problem (11) that $u_0(x) \in H^2(\mathbb{R}^3)$. Note that equality (6) can be easily derived by applying the $\sqrt{-\Delta}$ operator to both sides of (11) and using identity (14) for the function $f(x)$. Let us seek the resulting solution of the nonlinear equation (2) as

$$u(x) = u_0(x) + u_p(x). \tag{7}$$

Evidently, we obtain the perturbative equation

$$\sqrt{-\Delta}u_p = \varepsilon \int_{\mathbb{R}^3} \mathcal{K}(x-y)g(u_0(y) + u_p(y))dy. \tag{8}$$

Let us introduce a closed ball in the Sobolev space

$$B_\rho := \{u(x) \in H^2(\mathbb{R}^3) \mid \|u\|_{H^2(\mathbb{R}^3)} \le \rho\}, \quad 0 < \rho \le 1. \tag{9}$$

We seek the solution of (8) as the fixed point of the auxiliary nonlinear problem

$$\sqrt{-\Delta}u = \varepsilon \int_{\mathbb{R}^3} \mathcal{K}(x-y)g(u_0(y) + v(y))dy \tag{10}$$

in ball (9). For a given function $v(y)$ this is an equation with respect to $u(x)$. The left side of (10) contains the operator without Fredholm property $\sqrt{-\Delta}$: $H^1(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$. Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$, such that this operator has no bounded inverse. The analogous situation appeared in works [25] and [27] but as distinct from the present article, the problems treated there required orthogonality conditions. The fixed point technique was used in [20] to estimate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger operator involved in the nonlinear problem possessed the Fredholm property (see Assumption 1 of [20], also [9]). Let us define the interval on the real line

$$I := [-c_e\|u_0\|_{H^2(\mathbb{R}^3)} - c_e, \ c_e\|u_0\|_{H^2(\mathbb{R}^3)} + c_e]. \tag{11}$$

We make the following assumption on the nonlinear part of problem (2).

Assumption 1.2. *Let $g(s) : \mathbb{R} \to \mathbb{R}$, such that $g(0) = 0$ and $g'(0) = 0$. We also assume that $g(s) \in C_2(\mathbb{R})$, such that*

$$a_2 := sup_{s\in I}|g''(s)| > 0.$$

Evidently $a_1 := sup_{s \in I}|g'(s)| > 0$ as well, otherwise the function $g(s)$ will be constant on the interval I and a_2 vanishes. For instance, $g(s) = s^2$ clearly satisfies the assumption above. When $g(s) = s^n$, $n > 1$ it means that cells stimulate proliferation by the surrounding cells by direct cell-cell contact or by some signaling molecules. Such effects are known, in particular in cancer studies.

Let us introduce the operator T_g, such that $u = T_g v$, where u is a solution of equation (10). Our main statement is as follows.

THEOREM 1.3. *Let Assumptions* 1.1 *and* 1.2 *hold. Then for any* $0 < \rho \leq 1$ *there exists* $\varepsilon* > 0$ *such that equation (*10*) defines the map* $T_g : B_\rho \to B_\rho$*, which is a strict contraction for all* $0 < \varepsilon < \varepsilon*$*. The unique fixed point* $u_p(x)$ *of the map* T_g *is the only solution of equation (*8*) in* B_ρ*.*

Apparently the resulting solution $u(x)$ of problem (2) will be nontrivial due to the fact that the source term $f(x)$ is nontrivial and $g(0) = 0$ as assumed. We make use of the following elementary lemma.

LEMMA 1.4. *Consider the function* $\varphi(R) := \alpha R + \dfrac{\beta}{R^2}$ *for* $R \in (0, +\infty)$*, where the constants* $\alpha, \beta > 0$*. It attains the minimal value at* $R^* = \left(\dfrac{2\beta}{\alpha}\right)^{\frac{1}{3}}$*, which is given by* $\varphi(R^*) = \dfrac{3}{2^{\frac{2}{3}}}\alpha^{\frac{2}{3}}\beta^{\frac{1}{3}}$*.*

We proceed to the proof of our main result.

2. The existence of the perturbed solution (Proof of Theorem 1.3)

PROOF. Let us choose arbitrarily $v(x) \in B_\rho$ and denote the term involved in the integral expression in right side of equation (10) as

$$G(x) := g(u_0 + v).$$

We apply the standard Fourier transform (15) to both sides of problem (10) and arrive at

$$\widehat{u}(p) = \varepsilon(2\pi)^{\frac{3}{2}}\frac{\widehat{\mathcal{K}}(p)\widehat{G}(p)}{|p|}.$$

Hence for the norm we obtain

$$\|u\|^2_{L^2(\mathbb{R}^3)} = (2\pi)^3\varepsilon^2\int_{\mathbb{R}^3}\frac{|\widehat{\mathcal{K}}(p)|^2|\widehat{G}(p)|^2}{p^2}dp. \tag{1}$$

As distinct from works [25] and [27] involving the standard Laplacian operator in the diffusion term, here we do not try to control the norm

$$\left\|\frac{\widehat{\mathcal{K}}(p)}{|p|}\right\|_{L^\infty(\mathbb{R}^3)}.$$

Let us estimate the right side of (1) using (16) with $R>0$ as

$$(2\pi)^3\varepsilon^2\int_{|p|\le R}\frac{|\widehat{\mathcal{K}}(p)|^2|\widehat{G}(p)|^2}{p^2}dp+(2\pi)^3\varepsilon^2\int_{|p|>R}\frac{|\widehat{\mathcal{K}}(p)|^2|\widehat{G}(p)|^2}{p^2}dp\le$$

$$\le\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)}\left\{\frac{1}{2\pi^2}\|G(x)\|^2_{L^1(\mathbb{R}^3)}R+\frac{1}{R^2}\|G(x)\|^2_{L^2(\mathbb{R}^3)}\right\}. \tag{2}$$

Since $v(x)\in B_\rho$, we have

$$\|u_0+v\|_{L^2(\mathbb{R}^3)}\le\|u_0\|_{H^2(\mathbb{R}^3)}+1$$

and the Sobolev embedding (5) yields

$$|u_0+v|\le c_e\|u_0\|_{H^2(\mathbb{R}^3)}+c_e.$$

The formula $G(x)=\displaystyle\int_0^{u_0+v}g'(s)ds$ with the interval I defined in (11) implies

$$|G(x)|\le sup_{s\in I}|g'(s)||u_0+v|=a_1|u_0+v|.$$

Thus

$$\|G(x)\|_{L^2(\mathbb{R}^3)}\le a_1\|u_0+v\|_{L^2(\mathbb{R}^3)}\le a_1(\|u_0\|_{H^2(\mathbb{R}^3)}+1).$$

Obviously, $G(x)=\displaystyle\int_0^{u_0+v}ds\Big[\int_0^s g''(t)dt\Big]$. Hence, we arrive at

$$|G(x)|\le\frac{1}{2}sup_{t\in I}|g''(t)||u_0+v|^2=\frac{a_2}{2}|u_0+v|^2,$$

$$\|G(x)\|_{L^1(\mathbb{R}^3)}\le\frac{a_2}{2}\|u_0+v\|^2_{L^2(\mathbb{R}^3)}\le\frac{a_2}{2}(\|u_0\|_{H^2(\mathbb{R}^3)}+1)^2.$$

Thus, we obtain the upper bound for the right side of (2) as

$$\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)}(\|u_0\|_{H^2(\mathbb{R}^3)}+1)^2\left\{\frac{a_2^2}{8\pi^2}(\|u_0\|_{H^2(\mathbb{R}^3)}+1)^2R+\frac{a_1^2}{R^2}\right\}$$

with $R\in(0,+\infty)$. Lemma 1.4 yields the minimal value of the expression above. Thus

$$\|u\|^2_{L^2(\mathbb{R}^3)}\le\frac{3}{2^{\frac{2}{3}}4\pi^{\frac{4}{3}}}\varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)}(\|u_0\|_{H^2(\mathbb{R}^3)}+1)^{3\frac{1}{3}}a_1^{\frac{2}{3}}a_2^{\frac{4}{3}}. \tag{3}$$

Evidently, (10) implies

$$-\Delta u = \varepsilon\sqrt{-\Delta}\int_{\mathbb{R}^3}\mathcal{K}(x-y)G(y)dy$$

and

$$\nabla G(x) = g'(u_0+v)(\nabla u_0+\nabla v).$$

We will use the identity

$$g'(u_0+v) = \int_0^{u_0+v} g''(s)ds.$$

Sobolev embedding (5) yields

$$|g'(u_0+v)| \le sup_{s\in I}|g''(s)||u_0+v| \le a_2c_e(\|u_0\|_{H^2(\mathbb{R}^3)}+1).$$

The following inequality can be trivially obtained using the standard Fourier transform, namely

$$\|\nabla u\|_{L^2(\mathbb{R}^3)} \le \|u\|_{H^2(\mathbb{R}^3)}. \tag{4}$$

Then we arrive at

$$\|\Delta u\|^2_{L^2(\mathbb{R}^3)} \le \varepsilon^2\|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)}a_2^2c_e^2(\|u_0\|_{H^2(\mathbb{R}^3)}+1)^4. \tag{5}$$

The definition of the norm (4) along with estimates (3) and (5) imply

$$\|u\|_{H^2(\mathbb{R}^3)} \le \varepsilon\|\mathcal{K}\|_{L^1(\mathbb{R}^3)}(\|u_0\|_{H^2(\mathbb{R}^3)}+1)^2a_2^{\frac{2}{3}}\sqrt{\frac{3}{2^{\frac{2}{3}}4\pi^{\frac{4}{3}}}a_1^{\frac{2}{3}}+a_2^{\frac{2}{3}}c_e^2} \le \rho \tag{6}$$

for all positive values of ε small enough. Thus $u(x)\in B_\rho$ as well. If for some $v(x)\in B_\rho$ there exist two solutions $u_{1,2}(x)\in B_\rho$ of problem (10), their difference $w(x) := u_1(x)-u_2(x)\in L^2(\mathbb{R}^3)$ satisfies

$$\sqrt{-\Delta}w = 0.$$

Since the operator $\sqrt{-\Delta}$ does not have nontrivial square integrable zero modes, $w(x)=0$ a.e. in $\mathbb{R}^3$. Therefore, equation (10) defines a map $T_g : B_\rho \to B_\rho$ for $\varepsilon > 0$ sufficiently small.

The goal is to prove that this map is a strict contraction. We choose arbitrarily $v_{1,2}(x)\in B_\rho$. The argument above yields $u_{1,2} = T_g v_{1,2}\in B_\rho$ as well. (10) gives us

$$\sqrt{-\Delta}u_1 = \varepsilon\int_{\mathbb{R}^3}\mathcal{K}(x-y)g(u_0(y)+v_1(y))dy, \tag{7}$$

$$\sqrt{-\Delta}u_2 = \varepsilon\int_{\mathbb{R}^3}\mathcal{K}(x-y)g(u_0(y)+v_2(y))dy. \tag{8}$$

Let us define

$$G_1(x) := g(u_0 + v_1), \quad G_2(x) := g(u_0 + v_2)$$

and apply the standard Fourier transform (15) to both sides of problems (7) and (8). We obtain

$$\widehat{u_1}(p) = \varepsilon (2\pi)^{\frac{3}{2}} \frac{\widehat{\mathcal{K}}(p)\widehat{G_1}(p)}{|p|}, \quad \widehat{u_2}(p) = \varepsilon (2\pi)^{\frac{3}{2}} \frac{\widehat{\mathcal{K}}(p)\widehat{G_2}(p)}{|p|}.$$

Evidently

$$\|u_1 - u_2\|^2_{L^2(\mathbb{R}^3)} = \varepsilon^2 (2\pi)^3 \int_{\mathbb{R}^3} \frac{|\widehat{\mathcal{K}}(p)|^2 |\widehat{G_1}(p) - \widehat{G_2}(p)|^2}{|p|^2} dp.$$

Clearly, it can be bounded from above by means of (16) by

$$\varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)} \left\{ \frac{1}{2\pi^2} \|G_1(x) - G_2(x)\|^2_{L^1(\mathbb{R}^3)} R + \|G_1(x) - G_2(x)\|^2_{L^2(\mathbb{R}^3)} \frac{1}{R^2} \right\}$$

with $R \in (0, +\infty)$. Let us use the equality

$$G_1(x) - G_2(x) = \int_{u_0+v_2}^{u_0+v_1} g'(s) ds.$$

Thus

$$|G_1(x) - G_2(x)| \le sup_{s \in I} |g'(s)| |v_1 - v_2| = a_1 |v_1 - v_2|.$$

Therefore

$$\|G_1(x) - G_2(x)\|_{L^2(\mathbb{R}^3)} \le a_1 \|v_1 - v_2\|_{L^2(\mathbb{R}^3)} \le a_1 \|v_1 - v_2\|_{H^2(\mathbb{R}^3)}.$$

Evidently,

$$G_1(x) - G_2(x) = \int_{u_0+v_2}^{u_0+v_1} ds \Big[\int_0^s g''(t) dt \Big].$$

We derive the upper bound for $G_1(x) - G_2(x)$ in the absolute value as

$$\frac{1}{2} sup_{t \in I} |g''(t)| |(v_1 - v_2)(2u_0 + v_1 + v_2)| = \frac{a_2}{2} |(v_1 - v_2)(2u_0 + v_1 + v_2)|.$$

By means of the Schwarz inequality we estimate the norm $\|G_1(x) - G_2(x)\|_{L^1(\mathbb{R}^3)}$ from above by

$$\frac{a_2}{2} \|v_1 - v_2\|_{L^2(\mathbb{R}^3)} \|2u_0 + v_1 + v_2\|_{L^2(\mathbb{R}^3)} \le a_2 \|v_1 - v_2\|_{H^2(\mathbb{R}^3)} (\|u_0\|_{H^2(\mathbb{R}^3)} + 1).$$

Thus we arrive at the upper bound for the norm $\|u_1(x) - u_2(x)\|^2_{L^2(\mathbb{R}^3)}$ given by

$$\varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)} \|v_1 - v_2\|^2_{H^2(\mathbb{R}^3)} \Big\{ \frac{{a_2}^2}{2\pi^2} (\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^2 R + \frac{{a_1}^2}{R^2} \Big\}.$$

Lemma 4 enables us to minimize the expression above over $R > 0$ to obtain that $\|u_1(x) - u_2(x)\|^2_{L^2(\mathbb{R}^3)}$ is estimated from above by

$$\varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)} \|v_1 - v_2\|^2_{H^2(\mathbb{R}^3)} \frac{3}{2^{\frac{4}{3}}} \frac{a_2^{\frac{4}{3}}}{\pi^{\frac{4}{3}}} (\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^{\frac{4}{3}} a_1^{\frac{2}{3}}. \tag{9}$$

(7) and (8) imply that

$$-\Delta(u_1 - u_2) = \varepsilon\sqrt{-\Delta} \int_{\mathbb{R}^3} \mathcal{K}(x-y)[g(u_0(y) + v_1(y)) - g(u_0(y) + v_2(y))]dy.$$

Thus

$$\|\Delta(u_1 - u_2)\|^2_{L^2(\mathbb{R}^3)} \leq \varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)} \|\nabla g(u_0 + v_1) - \nabla g(u_0 + v_2)\|^2_{L^2(\mathbb{R}^3)}.$$

We express $\nabla g(u_0 + v_1) - \nabla g(u_0 + v_2)$ as

$$g'(u_0 + v_1)(\nabla u_0 + \nabla v_1) - g'(u_0 + v_2)(\nabla u_0 + \nabla v_2) =$$

$$= (\nabla u_0 + \nabla v_1) \int_{u_0+v_2}^{u_0+v_1} g''(s)ds + (\nabla v_1 - \nabla v_2) \int_0^{u_0+v_2} g''(s)ds.$$

This yields the estimate from above for $|\nabla g(u_0 + v_1) - \nabla g(u_0 + v_2)|$ as

$$sup_{s\in I}|g''(s)||v_1 - v_2||\nabla u_0 + \nabla v_1| + sup_{s\in I}|g''(s)||u_0 + v_2||\nabla v_1 - \nabla v_2|.$$

This expression can be trivially bounded from above by means of the Sobolev embedding (5) by

$$a_2 c_e \|v_1 - v_2\|_{H^2(\mathbb{R}^3)} |\nabla u_0 + \nabla v_1| + a_2 c_e \|u_0 + v_2\|_{H^2(\mathbb{R}^3)} |\nabla v_1 - \nabla v_2|.$$

Hence, by virtue of (4) for $v_{1,2} \in B_\rho$ we derive the upper bound for the norm $\|\Delta(u_1 - u_2)\|^2_{L^2(\mathbb{R}^3)}$ as

$$4\varepsilon^2 \|\mathcal{K}\|^2_{L^1(\mathbb{R}^3)} a_2^2 c_e^2 (\|u_0\|_{H^2(\mathbb{R}^3)} + 1)^2 \|v_1 - v_2\|^2_{H^2(\mathbb{R}^3)}. \tag{10}$$

By means of inequalities (9) and (10) the norm $\|u_1 - u_2\|_{H^2(\mathbb{R}^3)}$ is estimated from above by

$$\varepsilon \|\mathcal{K}\|_{L^1(\mathbb{R}^3)} (\|u_0\|_{H^2(\mathbb{R}^3)} + 1) a_2^{\frac{2}{3}} \left[\frac{3}{2^{\frac{4}{3}}} \frac{a_1^{\frac{2}{3}}}{\pi^{\frac{4}{3}}} + 4 a_2^{\frac{2}{3}} c_e^2 \right]^{\frac{1}{2}} \|v_1 - v_2\|_{H^2(\mathbb{R}^3)}.$$

Therefore, the map $T_g : B_\rho \to B_\rho$ defined by equation (10) is a strict contraction for all values of $\varepsilon > 0$ sufficiently small. Its unique fixed point $u_p(x)$ is the only solution of problem (8) in B_ρ. It can be observed that via (6) we have $u_p \to 0$ as $\varepsilon \to 0$ in the $H^2(\mathbb{R}^3)$ norm, which emphasizes the fact that it is a perturbative result. The resulting $u(x) \in H^2(\mathbb{R}^3)$ given by (7) is a solution of equation (2). Note that $v_2(x) = 0$ in $\mathbb{R}^3$ would not necessarily imply that $u_2(x) = 0$ in $\mathbb{R}^3$ as well for the contraction argument above.

$\square$

3. Auxiliary results

First we derive the solvability conditions for the following linear Poisson equation

$$\sqrt{-\Delta} u = f(x), \quad x \in \mathbb{R}^d, \quad d \in \mathbb{N}. \tag{11}$$

Our technical result is as follows.

LEMMA 3.1. *Let $f(x) \in L^2(\mathbb{R}^d), d \in \mathbb{N}$.*
1) When $d = 1$ and in addition $|x|f(x) \in L^1(\mathbb{R})$, equation (11) admits a unique solution $u(x) \in H^1(\mathbb{R})$ if and only if the orthogonality condition

$$\int_{-\infty}^{\infty} f(x)dx = 0 \tag{12}$$

holds.

2) When $d = 2$ and additionally $|x|f(x) \in L^1(\mathbb{R}^2)$, problem (11) possesses a unique solution $u(x) \in H^1(\mathbb{R}^2)$ if and only if the orthogonality relation

$$\int_{\mathbb{R}^2} f(x)dx = 0 \tag{13}$$

holds.

3) When $d \geq 3$ and in addition $f(x) \in L^1(\mathbb{R}^d)$, equation (11) has a unique solution $u(x) \in H^1(\mathbb{R}^d)$.

PROOF. Let us first address the uniqueness of solutions for problem (11). Suppose $u_{1,2}(x) \in H^1(\mathbb{R}^d)$ both satisfy equation (11). Then their difference $w(x) := u_1(x) - u_2(x)$ solves the homogeneous problem

$$\sqrt{-\Delta} w = 0.$$

Since the operator $\sqrt{-\Delta}$ in the the whole space does not have nontrivial square integrable zero modes, $w(x)$ vanishes a.e. in $\mathbb{R}^d$. Note that it would be sufficient to establish only the square integrability for the solution of (11). Indeed, we have a trivial identity

$$\|\sqrt{-\Delta} u\|^2_{L^2(\mathbb{R}^d)} = \|\nabla u\|^2_{L^2(\mathbb{R}^d)}. \tag{14}$$

Since the source term $f(x) \in L^2(\mathbb{R}^d)$ as assumed, we arrive at $\nabla u \in L^2(\mathbb{R}^d)$, such that $u(x) \in H^1(\mathbb{R}^d)$ as well. We will use the standard Fourier transform

$$\widehat{f}(p) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x)e^{-ipx}dx, \quad d \in \mathbb{N}. \tag{15}$$

Clearly, we have the estimate

$$\|\widehat{f}(p)\|_{L^\infty(\mathbb{R}^d)} \le \frac{1}{(2\pi)^{\frac{d}{2}}}\|f(x)\|_{L^1(\mathbb{R}^d)}. \tag{16}$$

Let us apply (15) to both sides of equation (11). We obtain

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|},$$

such that the norm can be expressed as

$$\|u\|^2_{L^2(\mathbb{R}^d)} = \int_{|p|\le 1} \frac{|\widehat{f}(p)|^2}{|p|^2}dp + \int_{|p|>1} \frac{|\widehat{f}(p)|^2}{|p|^2}dp. \tag{17}$$

Evidently, the second term in the right side of (17) can be estimated from above by $\|f\|^2_{L^2(\mathbb{R}^d)} < \infty$ as assumed. Let us estimate the first term in the right side of (17) in dimension $d = 1$, using the identity

$$\widehat{f}(p) = \widehat{f}(0) + \int_0^p \frac{d\widehat{f}(s)}{ds}ds.$$

Clearly, via definition (15)

$$\left|\frac{d\widehat{f}(p)}{dp}\right| \le \frac{1}{\sqrt{2\pi}}\||x|f\|_{L^1(\mathbb{R})} < \infty$$

due to one of our assumptions. Therefore,

$$\left|\frac{\int_0^p \frac{d\widehat{f}(s)}{ds}ds}{|p|}\chi_{\{p\in\mathbb{R}\ |\ |p|\le 1\}}\right| \le \frac{1}{\sqrt{2\pi}}\||x|f\|_{L^1(\mathbb{R})}\chi_{\{p\in\mathbb{R}\ |\ |p|\le 1\}} \in L^2(\mathbb{R}).$$

Here and further down χ_A stands for the characteristic function of a set $A \in \mathbb{R}^d$. The remaining term $\frac{\widehat{f}(0)}{|p|}\chi_{\{p\in\mathbb{R}\ |\ |p|\le 1\}}$ belongs to $L^2(\mathbb{R})$ if and only if $\widehat{f}(0)$ vanishes, which gives us orthogonality relation (12) in dimension $d = 1$.

Then we turn our attention to the case of dimension $d = 2$. Let us use the formula

$$\widehat{f}(p) = \widehat{f}(0) + \int_0^{|p|} \frac{\partial \widehat{f}}{\partial s}(s,\theta)ds,$$

where θ stands for the angle variable on the circle. Clearly, definition (15) yields

$$\left|\frac{\partial \widehat{f}}{\partial |p|}\right| \le \frac{1}{2\pi}\||x|f\|_{L^1(\mathbb{R}^2)} < \infty$$

as assumed. Thus

$$\left|\frac{\int_0^{|p|}\frac{\partial\widehat{f}(s,\theta)}{\partial s}ds}{|p|}\chi_{\{p\in\mathbb{R}^2\ |\ |p|\leq 1\}}\right|\leq\frac{1}{2\pi}\||x|f\|_{L^1(\mathbb{R}^2)}\chi_{\{p\in\mathbb{R}^2\ |\ |p|\leq 1\}}\in L^2(\mathbb{R}^2).$$

Finally, the term $\dfrac{\widehat{f}(0)}{|p|}\chi_{\{p\in\mathbb{R}^2\ |\ |p|\leq 1\}}\in L^2(\mathbb{R}^2)$ if and only if $\widehat{f}(0)=0$, such that we obtain orthogonality condition (13) for dimension $d=2$.

To complete the proof of the lemma, it remains to study the case of higher dimensions $d\geq 3$. By virtue of inequality (16), we easily estimate the first term in the right side of (17) by

$$\frac{1}{(2\pi)^d}\|f(x)\|^2_{L^1(\mathbb{R}^d)}|S_d|\frac{1}{d-2}<\infty$$

due to one of our assumptions. Here S_d denotes the unit sphere in the space of d dimensions centered at the origin and $|S_d|$ stands for its Lebesgue measure. □

Note that in dimensions $d\geq 3$ under the assumptions given above no orthogonality conditions are needed to solve the linear Poisson equation (11) in $H^1(\mathbb{R}^d)$.

Let us show that it is possible to incorporate a shallow, short-range potential into the linear Poisson equation considered above and generalize the result of Lemma 3.1. No orthogonality relations will be required in Lemma 3.3 below as well. Consider the following equation

$$\sqrt{-\Delta+V(x)}u=f(x),\quad x\in\mathbb{R}^3,\tag{18}$$

with the operator $\sqrt{-\Delta+V(x)}$ well defined via the spectral calculus, since under our assumptions the operator $-\Delta+V(x)$ on $L^2(\mathbb{R}^3)$ is nonnegative as discussed below.

Assumption 3.2. *The potential function $V(x):\mathbb{R}^3\to\mathbb{R}$ satisfies the estimate*

$$|V(x)|\leq\frac{C}{1+|x|^{3.5+\varepsilon}}$$

with some $\varepsilon>0$ and $x\in\mathbb{R}^3$ a.e. such that

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|V\|^{\frac{1}{9}}_{L^\infty(\mathbb{R}^3)}\|V\|^{\frac{8}{9}}_{L^{\frac{4}{3}}(\mathbb{R}^3)}<1\quad\text{and}\quad\sqrt{c_{HLS}}\|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}<4\pi.$$

This is analogous to Assumption 1.1 of [21] under which by virtue of Lemma 2.3 of [21] our Schrödinger operator $-\Delta+V(x)$ is self-adjoint and unitarily equivalent to $-\Delta$ on $L^2(\mathbb{R}^3)$ via the wave operators. Thus the essential spectrum of $\sqrt{-\Delta+V(x)}:H^1(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$ fills the nonnegative semi-axis $[0,+\infty)$. Hence such operator does not have a bounded inverse and therefore it fails to satisfy the Fredholm property. Here C stands for a finite, positive constant and c_{HLS} for the constant in the Hardy-Littlewood-Sobolev inequality

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f_1(x)f_1(y)}{|x-y|^2}dxdy\right| \leq c_{HLS}\|f_1\|^2_{L^{\frac{3}{2}}(\mathbb{R}^3)}, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3)$$

given on p.98 of [12]. The functions of the continuous spectrum of our Schrödinger operator satisfy

$$(-\Delta + V(x))\varphi_k(x) = k^2\varphi_k(x), \quad k \in \mathbb{R}^3,$$

in the integral formulation the Lippmann-Schwinger equation (see e.g. p.98 of [19])

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i|k||x-y|}}{|x-y|}(V\varphi_k)(y)dy \tag{19}$$

and the orthogonality relations

$$\int_{\mathbb{R}^3}\varphi_k(x)\bar{\varphi}_q(x)dx = \delta(k-q), \quad k, q \in \mathbb{R}^3.$$

They form a complete system in $L^2(\mathbb{R}^3)$. Let us denote by tilde the generalized Fourier transform with respect to these functions, such that

$$\tilde{f}(k) := \int_{\mathbb{R}^3} f(x)\bar{\varphi}_k(x)dx, \quad k \in \mathbb{R}^3. \tag{20}$$

The integral operator involved in the right side of equation (19) is

$$(Q\varphi)(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{i|k||x-y|}}{|x-y|}(V\varphi)(y)dy, \quad \varphi \in L^\infty(\mathbb{R}^3).$$

We consider $Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$. By virtue of Lemma 2.1 of [21] under Assumption 6 above on the scalar potential we have $\|Q\|_\infty < 1$. Moreover, this norm is bounded above by the quantity, which is independent of the wave vector k and can be expressed in terms of the appropriate $L^p(\mathbb{R}^3)$ norms of $V(x)$. Corollary 2.2 of [21] yields the estimate

$$|\tilde{f}(k)| \leq \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{1-\|Q\|_\infty}\|f\|_{L^1(\mathbb{R}^3)}. \tag{21}$$

We have the following statement.

Lemma 3.3. *Let the potential $V(x)$ satisfy Assumption* 3.2 *and $f(x) \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. Then equation (*18*) has a unique solution $u(x) \in H^1(\mathbb{R}^3)$.*

Proof. Let us first suppose that problem (18) has two solutions $u_{1,2}(x) \in H^1(\mathbb{R}^3)$. Then their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^3)$ solves the homogeneous equation

$$\sqrt{-\Delta + V(x)}w = 0,$$

which cannot have nontrivial square integrable solutions due to the fact that our self-adjoint operator $-\Delta + V(x)$ is unitarily equivalent to $-\Delta$ on $L^2(\mathbb{R}^3)$. Hence $w(x)$ vanishes a.e. in $\mathbb{R}^3$.

Let us apply the generalized Fourier transform (20) to both sides of equation (18) to obtain

$$\tilde{u}(k) = \frac{\tilde{f}(k)}{|k|}, \quad k \in \mathbb{R}^3.$$

This enables us to express the norm as

$$\|u\|^2_{L^2(\mathbb{R}^3)} = \int_{|k|\leq 1} \frac{|\tilde{f}(k)|^2}{k^2} dk + \int_{|k|>1} \frac{|\tilde{f}(k)|^2}{k^2} dk. \tag{22}$$

Clearly, the second term in the right side of (22) can be bounded from above by $\|f\|^2_{L^2(\mathbb{R}^3)} < \infty$ as assumed. Let us use (21) to estimate from above the first term in the right side of (22) as

$$\frac{1}{2\pi^2} \frac{1}{(1 - \|Q\|_\infty)^2} \|f\|^2_{L^1(\mathbb{R}^3)} < \infty$$

as well. Hence $u(x) \in L^2(\mathbb{R}^3)$. A trivial calculation using (18) yields

$$\|f\|^2_{L^2(\mathbb{R}^3)} = \|\nabla u\|^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} V(x)|u(x)|^2 dx.$$

Since $f(x)$ is square integrable and the scalar potential $V(x)$ is bounded as assumed, we have $\nabla u(x) \in L^2(\mathbb{R}^3)$ as well, such that the solution $u(x) \in H^1(\mathbb{R}^3)$. $\square$

Acknowledgements. Valuable discussions with A.Nepomnyashchy are gratefully acknowledged.

References

[1] G.L. Alfimov, E.V. Medvedeva, D.E. Pelinovsky, *Wave Systems with an Infinite Number of Localized Traveling Waves*, Phys. Rev. Lett., **112** (2014), 054103, 5 pp.

[2] C. Amrouche, V. Girault, J. Giroire, *Dirichlet and Neumann exterior problems for the n-dimensional Laplace operator. An approach in weighted Sobolev spaces*, J. Math, Pures Appl., **76** (1997), pp. 55–81.

[3] C. Amrouche, F. Bonzom, Mixed exterior Laplace's problem, J. Math. Anal. Appl., **338** (2008), pp. 124–140.

[4] P. Bolley, T.L. Pham, *Propriété d'indice en théorie Holderienne pour des opérateurs différentiels elliptiques dans R^n*, J. Math. Pures Appl., **72** (1993), pp. 105–119.

[5] P. Bolley, T.L. Pham, *Propriété d'indice en théorie Hölderienne pour le problème extérieur de Dirichlet*, Comm. Partial Differential Equations, **26** (2001), No. 1-2, pp. 315–334.

[6] N. Benkirane, *Propriété d'indice en théorie Holderienne pour des opérateurs elliptiques dans R^n*, CRAS, **307**, Série I (1988), pp. 577–580.

[7] X. Cabre, J.-M. Roquejoffre, *The influence of fractional diffusion in Fisher-KPP equations*, Comm. Math. Phys., **320** (2013), No. 3, pp. 679–722.

[8] L. Caffarelli, L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations, **32** (2007), No. 7–9, pp. 1245–1260.

[9] S. Cuccagna, D. Pelinovsky, V. Vougalter, *Spectra of positive and negative energies in the linearized NLS problem*, Comm. Pure Appl. Math., **58** (2005), No. 1, pp. 1–29.

[10] A. Ducrot, M. Marion, V. Volpert, *Systemes de réaction-diffusion sans propriété de Fredholm*, CRAS, **340** (2005), pp. 659–664.

[11] A. Ducrot, M. Marion, V. Volpert, *Reaction-diffusion problems with non Fredholm operators*, Advances Diff. Equations, **13** (2008), No. 11-12, pp. 1151–1192.

[12] E. Lieb, M. Loss, *Analysis. Graduate Studies in Mathematics*, **14**, American Mathematical Society, Providence (1997).

[13] T. Solomon, E. Weeks, H. Swinney, *Observation of anomalous diffusion and Lévy flights in a two-dimensional rotating flow*, Phys. Rev. Lett. 71 (1993), pp. 3975–3978.

[14] B. Carreras, V. Lynch, G. Zaslavsky, *Anomalous diffusion and exit time distribution of particle tracers in plasma turbulence model*, Phys. Plasmas, 8 (2001), pp. 5096–5103.

[15] P. Manandhar, J. Jang, G.C. Schatz, M.A. Ratner, S. Hong, *Anomalous surface diffusion in nanoscale direct deposition processes*, Phys. Rev. Lett., 90 (2003), pp. 4043–4052.

[16] J. Sancho, A. Lacasta, K. Lindenberg, I. Sokolov, A. Romero, *Diffusion on a solid surface: Anomalous is normal*, Phys. Rev. Lett., 92 (2004), 250601.

[17] H. Scher, E. Montroll, *Anomalous transit-time dispersion in amorphous solids*, Phys. Rev. B, 12 (1975), pp. 2455–2477.

[18] V. Volpert, *Elliptic partial differential equations. Volume 1. Fredholm theory of elliptic problems in unbounded domains*, Birkhauser, 2011.

[19] M. Reed, B. Simon, *Methods of modern mathematical physics, Volume III: Scattering theory*, Academic Press, 1979.

[20] V. Vougalter, *On threshold eigenvalues and resonances for the linearized NLS equation*, Math. Model. Nat. Phenom., **5** (2010), No. 4, pp. 448–469.

[21] V. Vougalter, V. Volpert, *Solvability conditions for some non-Fredholm operators*, Proc. Edinb. Math. Soc., **54** (2011), No.1, pp. 249–271.

[22] V. Vougalter, V. Volpert, *On the solvability conditions for the diffusion equation with convection terms,* Commun. Pure Appl. Anal., **11** (2012), No. 1, pp. 365–373.

[23] V. Vougalter, V. Volpert, *Solvability relations for some non Fredholm operators,* Int. Electron. J. Pure Appl. Math., **2** (2010), No. 1, pp. 75–83.

[24] V. Volpert, V. Vougalter, *On the solvability conditions for a linearized Cahn-Hilliard equation,* Rend. Istit. Mat. Univ. Trieste, **43** (2011), pp. 1–9.

[25] V. Vougalter, V. Volpert, *On the existence of stationary solutions for some non-Fredholm integro-differential equations*, Doc. Math., **16** (2011), pp. 561–580.

[26] V. Vougalter, V. Volpert. *Solvability conditions for a linearized Cahn-Hilliard equation of sixth order,* Math. Model. Nat. Phenom., **7** (2012), No. 2, pp. 146–154.

[27] V. Vougalter, V. Volpert. *Solvability conditions for some linear and nonlinear non-Fredholm elliptic problems,* Anal. Math. Phys., **2** (2012), No.4, pp. 473–496.

[28] V. Vougalter, V. Volpert. *Existence of stationary solutions for some nonlocal reaction-diffusion equations.* Dyn. Partial Differ.Equ., 12 (2015), No.1, pp. 43–51.

[29] V.A. Volpert, Y.Nec, A.A. Nepomnyashchy. *Exact solutions in front propagation problems with superdiffusion,* Phys. D, **239** (2010), No.3–4, pp. 134–144.

[30] V.A. Volpert, Y.Nec, A.A. Nepomnyashchy, *Fronts in anomalous diffusion-reaction systems,* Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., **371** (2013), No. 1982, 20120179, 18 pp.

Received submission date; revised revision date