Linear Forms in a Playful Universe

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Abstract – Instead of the axiom of choice, we assume that every set of reals has the Baire property. It is shown that under this condition the concept of slenderness known from the theory of abelian groups becomes meaningful for vector spaces.

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1. Introduction

Usually in mathematics the axiom of choice AC is assumed. One consequence of this axiom is that every vector space has a base. This means that each vector space is a direct sum of a number of copies of the underlying field. Thus all vector spaces have the same algebraic structure. This is the reason why the special case of vector spaces is uninteresting from a module-theoretical point of view.

In the theory of abelian groups there is a lot of structural diversity instead. An example is that for abelian groups infinite cartesian products on the one hand and infinite direct sums on the other hand are two fundamentally different structures. In particular, infinite cartesian products of abelian groups are far from being free, i.e. from having a basis. This is related to the concept of slenderness for abelian groups (cf. [3, 159 ff.]) and the fact that the ring \( \mathbb{Z} \) is a slender abelian group, discovered by Specker [12]. Slenderness for modules over arbitrary rings was studied by Lady [7].

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A ring \( R \) is called slender if \( R \) is slender as \( R \)-module. We give the definition for slenderness in section 2.

The basis theorem for vector spaces is the reason why fields fail to be slender. One can ask what the situation is in a set theory in which the axiom of choice does not apply. As an alternative to the axiom of choice, we focus our attention on the axiom of determinacy \( AD \) or its weaker variants. This was inspired by a paper by Felgner and Schulz [1]. The axiom \( AD \) was introduced by Mycielski and Steinhaus [8]. A well known conclusion is that every set of reals has the Baire property. Shelah proved that the theory \( ZF + DC + BP \) is equiconsistent to \( ZFC \) [11]. Here \( ZF \) is the Zermelo-Fran\‘kel set theory, \( ZFC \) is the Zermelo-Fran\‘kel set theory together with the axiom of choice, \( DC \) is the axiom of dependent choice and \( BP \) means that every set of reals has the Baire property. \( AD \) is more stringent than \( BP \) because \( AD \) implies the existence of measurable cardinals. For our purposes it is sufficient to assume \( BP \). Our result is that under this assumption, fields like \( \mathbb{R} \), \( \mathbb{Q} \) and also at most countable fields are slender rings.

The fact that a set of reals has the Baire property is related to the determinacy of so-called Bannach-Masur games. The definition of this infinite games was first given by Oxtoby [10]. We give a short outline in section 3 and refer to the explanations in the book of Jech [5, pp. 553-555].

The last section deals with the subgroup \( B \) of the Bear-Specker group, which consists of all bounded functions. N\‘obeling has shown that \( B \) is a free abelian group [9], while Specker previously came to the same result by assuming the continuum hypothesis [12]. In both articles, \( ZFC \) is assumed as usual. Felgner and Schulz showed that under the assumption of \( AD \) the group \( B \) fails to be free [1]. In section 5 we give results which indicate that under \( ZF + DC + BP \) the structure of \( B \) is more similar to that of the complete Bear-Specker group than that of a direct sum.

We use notations that are common in set theory. The set of natural numbers is denoted by \( \omega \).

### 2. On Slenderness

Let \( R \) be a ring. Then \( R^\omega \) is the \( R \)-module consisting of all functions from \( \omega \) into \( R \). For \( i \in \omega \) let \( e_i \) be the element of \( R^\omega \) for which \( e_i(j) = 1 \) if \( i = j \) and \( e_i(j) = 0 \) otherwise. It’s common to write some \( a \in R^\omega \) as the infinite sum \( \sum_{i \in \omega} a(i) e_i \). A \( R \)-module \( M \) is called slender, if for every homomorphism \( \varphi \) from \( R^\omega \) into \( M \) it is \( \varphi(e_i) = 0 \) for almost all \( i \in \omega \). It is not difficult to see that \( \varphi = 0 \) in the case that \( \varphi(e_i) = 0 \) for all \( i \in \omega \). This implies
that every homomorphism $\varphi$ from $R^\omega$ into a slender module $M$ is induced by a finite sequence $x_0, \ldots, x_{n-1}$ in $M$ such that $\varphi(a) = \sum_{i<n} a(i)x_i$. Hence $\text{hom}(R^\omega, M) \cong M^{<\omega}$, where $M^{<\omega}$ is the direct sum of countable many copies of $M$. For a module $M$ the dual module $M^*$ is the module hom$(M, R)$. It is easy to see that $(R^{<\omega})^* \cong R^\omega$. If $R$ is slender we also have the converse $(R^\omega)^* \cong R^{<\omega}$.

If $\prod_{i \in \omega} A_i$ is an arbitrary product of $R$-modules $A_i$ and $\varphi$ a homomorphism from $\prod_{i \in \omega} A_i$ into a slender $R$-module $M$, then there exists some $n \in \omega$ and homomorphisms $\varphi_i$ from $A_i$ into $M$ for $i < n$ such that $\varphi(a) = \sum_{i<n} \varphi_i(a_i)$ for all $a \in \prod_{i \in \omega} A_i$. It is a conclusion that a $R$-module is slender as a $R$-module, if it is slender as an abelian group. In 4 we show that under our set theoretical assumption $\mathbb{Q}$ and $\mathbb{R}$ are slender abelian groups. Hence $\mathbb{Q}$ and $\mathbb{R}$ are also slender as rings.

3. On Games

For a set $X$ and natural number $n \in \omega$ we denote by $X^n$ the set of all functions from $n$ to $X$. Furthermore $X^{<\omega}$ is the union $\bigcup_{n \in \omega} X^n$. Functions are identified with their graphs and hence for functions $f$ and $g$ notations like $f \subset g$ and $f \cup g$ make sense. An element $s$ of $X^{<\omega}$ is also viewed as the sequence $s = (s(0), \ldots, s(n-1))$. For two sequences $s$ and $t$ the concatenation $s \cdot t$ is defined in the natural way. If we have a sequence $s_n \in X^{<\omega}$ for each $n \in \omega$ we can build an infinite concatenation

$$a = s_0 \cdot s_1 \cdot s_2 \cdot \ldots$$

Here $a \in X^\omega$ is the union $\bigcup_{n \in \omega} a_n$ where $a_0 = s_0$ and $a_n = a_{n-1} \cdot s_n$.

It is common to regard $X^\omega$ also from a topological point of view. To do this, take the discrete topology on $X$ and build the product topology on $X^\omega$. In this topology the set $\{U_s : s \in X^{<\omega} \land s \subset a\}$ build an neighbour basis for an element $a \in X^\omega$, where $U_s$ is defined by $\{b \in X^\omega : s \subset b\}$ for $s \in X^{<\omega}$. Let us look again at the infinite concatenation $a$ we built above. In this situation we have $\{a\} = \bigcap_{n \in \omega} U_{a_n}$ where $a_n$ is defined as before.

From now on we assume that $X$ has at least two elements. For a set $A \subset X^{\omega}$ we define an infinite game $G^*_X(A)$ for two players as follows: The two players alternately choose finite sequences $s_n$ from $X^{<\omega}$ at step $n \in \omega$. Player I wins the game if the concatenation $a$ build like (1) is an element of $A$. Otherwise player II wins the game. Of course, if $A$ is small in a certain way, player II has a good chance to win the game. For example, it is not difficult to see that there is a winning strategy for player II if $A$ is at most
countable. The game is determined when there is a winning strategy either for player I or for player II.

You can easily modify a winning strategy for player I in the game \( G^{**}_X(A) \) and get a winning strategy for player II in game \( G^{**}_X(U_s \setminus A) \) when \( s \) is the starting sequence of player I in the first game. Here \( U_s \) is defined by \( \{ a \in X^\omega : s \subset a \} \). Similarly, a winning strategy for player II for a game \( G^{**}_X(U_s \setminus A) \) results in a winning strategy for player I for the game \( G^{**}_X(A) \).

In a topological space a subset \( A \) is called nowhere dense, if the interior of the closure of the set is empty. \( A \) is called meager, if \( A \) is a countable union of nowhere dense subsets. We say \( A \) has the Baire property, if there is an open set \( B \) such that \( A \triangle B = (A \setminus B) \cup (B \setminus A) \) is meager. Of course meager sets has the Baire property. If \( A \) has the Baire property and is not meager, then there is an \( s \) such that \( U_s \setminus A \) is meager.

It is known that player II has a winning strategy for the game \( G^{**}_X(A) \) exactly when \( A \) is meager and Player I has a winning strategy for \( G^{**}_X(A) \) if \( U_s \setminus A \) is meager. Hence if \( A \) has the Baire property, then \( G^{**}_X(A) \) is determined.

We are especially interested in the two cases in which \( X \) is \( \{0,1\} \) or \( \omega \). Then \( X^\omega \) is the Cantor space respectively Baire space. Both cases can be viewed as topological subspaces of the real line. So if we assume that every set of reals has the Baire property, all games \( G^{**}_X(A) \) for our two cases of \( X \) are determined. Our proofs in the further course of this paper follow the same pattern. We begin with a decomposition \( \bigcup_{n \in \mathbb{N}} A_n \) of the base set \( X^\omega \). Then one of the subsets \( A_n \) fails to be meager. Then for this \( A_n \), player I has a winning strategy.

4. The Main Result

Our first Theorem is about slenderness of abelian groups

**Theorem 4.1.** In the set theoretical setting \( ZF + DC + BP \) the following holds

a) Every linearly ordered abelian group is slender.

b) Every at most countable abelian group is slender.

**Proof.** For the proof of a) we start with a homomorphism \( \varphi \) from \( \mathbb{Z}^\omega \) to a linearly ordered abelian group \( G \). The domain \( \mathbb{Z}^\omega \) of \( \varphi \) is divided in the subsets: \( A_1 = \{ x \in \mathbb{Z}^\omega : \varphi(x) < 0 \} \), \( A_2 = \{ x \in \mathbb{Z}^\omega : \varphi(x) = 0 \} \) and \( A_3 = \{ x \in \mathbb{Z}^\omega : \varphi(x) > 0 \} \). By Lemma 2.1 player I has a winning strategy
for one of the three sets. Let us start with a winning strategy for $A_1$. The strategy gives the first move $s = \langle x_0, \ldots, x_k \rangle$ for player I. We will show that $\varphi(e_i) = 0$ for all $i > k$. Assuming $\varphi(e_m) \neq 0$ for some $m > k$ we choose $c \in \mathbb{Z}$ such that

$$\sum_{i=0}^{k} s_i \varphi(e_i) < c \varphi(e_m)$$

(2)

Now we play two games, both opened by player I with move $s_0$. In the first game we take as the first move of player II the sequence $s_1 = \langle 0, \ldots, 0, -c \rangle$ just with length $m - k$. After this player I makes his second move $s_2$ in the first game concordantly with the strategy. Unlike in the first game, player II responds in the second game with the concatenation of $s_1$ and $-s_2$, where $-s_2$ is formed from $s_2$ by changing the sign. From now on player II always plays the sequence $-s$ if player I has played $s$ in the other game before. So as a result of the two games we get two elements of $\mathbb{Z}^\omega$

$$a = s_0 \hat{s}_1 \hat{s}_2 \hat{s}_3 \hat{s}_4 \ldots$$

$$b = s_0 \hat{(s_1 \hat{s}_2)} \hat{-s_3} \hat{s}_4 \ldots$$

which are in $A_1$ because player I used his winning strategy for $A_1$. Thus

$$0 > \varphi(a) + \varphi(b) = \varphi(a + b) = \sum_{i=0}^{k} s_i \varphi(e_i) - c \varphi(e_m)$$

This is a contradiction to equation (2).

The proof for the cases that player I has a winning strategy for $A_2$ or for $A_3$ works analogously with the correct choice of $c$.

We proceed in a similar way for the proof of part b). Let $\varphi$ be an homomorphism from $\mathbb{Z}^\omega$ to an abelian group $G$ and $G$ is at most countable. For one of the subsets $A_g = \{ x \in \mathbb{Z}^\omega : \varphi(x) = g \}$ player I must have a winning strategy. As above we construct two sequences

$$a = s_0 \hat{s}_1 \hat{s}_2 \hat{s}_3 \hat{s}_4 \ldots$$

$$b = s_0 \hat{(s'_1 \hat{s}_2)} \hat{s'_3} \hat{s}_4 \ldots$$

where $s_1 = \langle 0, \ldots, 0, 1 \rangle$ and $s'_1 = \langle 0, \ldots, 0, 0 \rangle$ are of the same length $m + 1$. In the construction of $a$ the part $s_1$ is the move of player II and $s_2$...
the second move of player I. In the construction of \(b\) the first move of player II is \(s_1 \sim s_2\) and \(s_3\) is the second move of player I. We get

\[
0 = g - g = \varphi(a) - \varphi(b) = \varphi(a - b) = \varphi(e_{k+m})
\]

where \(k\) is the length of \(s_0\). Because we can choose \(m\) arbitrarily \(\varphi(e_i) = 0\) for almost all \(i \in \omega\).

Because subgroups of slender groups are slender and because exact sums of slender groups are slender, the arguments we have noted in 3 results in the following corollary.

**Corollary 4.2.** In the set theoretical setting \(ZF + DC + BP\) the following holds

a) Every subfield of the field of complex numbers is a slender ring.

b) Every at most countable field is a slender ring.

5. A Further Result

Let \(B\) be the subgroup of \(\Z^\omega\) which contains all functions \(a : \omega \to \Z\) for which \(a(\omega)\) is finite. An abelian group is called \(B\)-slender if for every homomorphism from \(B\) into \(G\) we have \(\varphi(e_i) = 0\) for almost all \(i \in \omega\).

**Theorem 5.1.** The assumption \(ZF + DC + BP\) implies that every at most countable abelian group is \(B\)-slender.

**Proof.** Unlike in the proof of 4.1 b), we define the sets \(A_g\) by \(\{x \in \{0, 1\}^\omega : \varphi(x) = g\}\). We can then proceed in the same way, since the construction of \(a\) and \(b\) does not leave the set \(\{0, 1\}^\omega\). □

The subgroup \(B\) is a special case of the so-called monotonic subgroups of \(\Z^\omega\). \(M\)-slenderness for monotonic subgroups \(M\) of \(\Z^\omega\) has been studied in detail by Fuchs and the author together with Göbel and Kolman \([2][13][4][6]\). However, ZFC has always been assumed and consequently, because of Nöbeling’s theorem, the trivial group 0 was the only \(B\)-slender group.

Now we ask whether we can extend theorem 5.1 analogous to theorem 4.1 a) to linearly ordered groups. The answer is no. For this we define a homomorphism from \(B\) to \(\R\) as follows. We take a absolute convergent series \(\sum_{i \in \omega} c_i\) in \(\R\) and define \(\varphi\) from \(B\) to \(\R\) by \(\varphi(a) = \sum_{i \in \omega} a(i)c_i\) for all \(a \in B\).
Theorem 5.2. We assume $ZF + DC + BP$. If $G$ is a $B$-slender abelian group an $\varphi$ an homomorphism from $B$ into $G$ with $\varphi(e_i) = 0$ for all $i \in \omega$, the $\varphi = 0$.

Proof. Note that the simple proof of the analogous statement for slenderness does not work for $B$-slenderness. Nevertheless the theorem is valid in our universe. We use the known fact that under $ZF + DC + BP$ every ultrafilter of $\omega$ is principal. Because this can be proven with the same method we have applied here, we will give a sketch of the proof.

In a well-known way, we understand subsets of $\omega$ as elements of $\{0,1\}^\omega$. Hence an ultrafilter $\mathcal{F}$ can viewed as a subset of $\{0,1\}^\omega$ and thus player I has a winning strategy either for the game $G^*_{\mathcal{F}}$, or for the game $G^*_\mathcal{F}$(\{0,1\}^\omega \setminus \mathcal{F})$. As in the previous proofs, we can use the winning strategy for the game to construct two functions $a$ and $b$ in $\{0,1\}^\omega$ that differ in almost all places. This defines two sets $A$ and $B$ for which $A \cap B$ and $(\omega \setminus A) \cap (\omega \setminus B)$ are finite. Because $A$ and $B$ are based on the winning strategy for the same of the two games, either $A$ and $B$ belong to $\mathcal{F}$ or $A$ and $B$ belong not to $\mathcal{F}$. But if $A$ and $B$ are not in $\mathcal{F}$ then $\omega \setminus A$ and $\omega \setminus B$ are in $\mathcal{F}$. Hence in both cases $\mathcal{F}$ contains a finite element. This shows that $\mathcal{F}$ must be a principal filter.

Now we start with the proof of the theorem and assume that $\varphi(a) \neq 0$ for some $a \in B$. For any $W \subseteq \omega$ we define $a|_W = \sum_{i \in W} a(i)e_i$. First we assume

$$(\forall W \subseteq \omega)(\varphi(a|_W) \neq 0 \Rightarrow (\exists U, V \subseteq W)(U \cap V = \emptyset \wedge \varphi(a|_U) \neq 0 \wedge \varphi(a|_V) \neq 0))$$

and lead this to a contradiction.

From this assumption we can use $DC$ to get a family $(U_j, V_j)_{j \in \omega}$ of pairs of subsets of $\omega$ such that $U_j \cap V_j = \emptyset$, $\varphi(a|_{U_j}) \neq 0$, $\varphi(a|_{V_j}) \neq 0$ and furthermore $U_{j+1} \subset U_j$ and $V_{j+1} \subset U_j$ for all $j \in \omega$. Notice that the family $(V_j)_{j \in \omega}$ is pairwise disjoint. Hereby we can construct a new homomorphism $\psi$ from $B$ into $G$ by defining $\psi(x) = \varphi(\sum_{j \in \omega} x(i)a|_{V_j})$ for $x \in B$. Because all $a|_{V_j}$ are parts of the same bounded function $a$ and $x$ also is a bounded function, the construction $\sum_{j \in \omega} x(i)a|_{V_j}$ is bounded as well. But now we have a contradiction to the $B$-slenderness of $G$, because $\psi(e_j) = \varphi(a|_{V_j}) \neq 0$ for all $j \in \omega$.

Consequently there must be some $W \subseteq \omega$ with $\varphi(a|_W) \neq 0$ and

$$(3) \quad (\forall U, V \subseteq W)(U \cap V = \emptyset \Rightarrow (\varphi(a|_U) = 0 \vee \varphi(a|_V) = 0))$$
Next we show that the set
\[ \mathcal{F} = \{ U \subseteq \omega : \varphi(a|_{W \cap U}) \neq 0 \} \]
is an ultrafilter. Therefore we have to prove
a) \( \emptyset \notin \mathcal{F} \)
b) If \( U \subseteq V \subseteq \omega \) and \( U \in \mathcal{F} \), then \( V \in \mathcal{F} \)
c) If \( U \in \mathcal{F} \) and \( V \in \mathcal{F} \), then \( U \cap V \in \mathcal{F} \)
d) For every \( U \subseteq \omega \) either \( U \in \mathcal{F} \) or \( \omega \setminus U \notin \mathcal{F} \)
a) is obvious. For b) notice that \( (W \cap U) \cap (W \cap (V \setminus U)) = \emptyset \) and\( \varphi(a|_{W \cap U}) \neq 0 \). Hence by (3) \( \varphi(a|_{W \cap (V \setminus U)}) = 0 \). Because
\[ a|_{W \cap V} = a|_{W \cap U} + a|_{W \cap (V \setminus U)} \]
we get \( \varphi(a|_{W \cap V}) \neq 0 \) and therefore \( V \in \mathcal{F} \). For c) we get \( \varphi(a|_{W \cap (V \setminus U)}) = 0 \) in the same way. Now we have
\[ a|_{W \cap V} = a|_{W \cap U \cap V} + a|_{W \cap (V \setminus U)} \]
and hence \( \varphi(a|_{W \cap U \cap V}) = \varphi(a|_{W \cap V}) \neq 0 \). This means that \( U \cap V \in \mathcal{F} \). At least d) is a direct consequence of \( \varphi(a|_W) \neq 0 \).

The assumption that \( \varphi(e_i) = 0 \) for all \( i \in \omega \) implies that \( \mathcal{F} \) fails to principal. \( \Box \)

A consequence of this theorem is that images of homomorphisms of \( B \) into \( B \)-slender abelian groups are finitely generated as we know it from slenderness too. This is required for the next corollary.

**Corollary 5.3.** Direct sums of \( B \)-slender groups are \( B \)-slender.

**Proof.** The proof is identical to that given in [3, p. 160] for slenderness. Let \( \varphi \) be a homomorphism into a sum \( \bigoplus_{i \in I} G_i \) of \( B \)-slender abelian groups \( G_i \). Then the image of \( \varphi \) is contained in a subgroup \( \bigoplus_{i \in I'} G'_i \) of the full sum, where \( G'_i \) is a finitely generated subgroup of \( G_i \) and \( I' \) is at most countable. Thus the image of \( \varphi \) is countable and therefore also \( B \)-slender. Hence \( \varphi(e_i) = 0 \) for almost all \( i \in \omega \). \( \Box \)

In conclusion, it remains the open question whether any of the statements we have proved with the axiom BP is equivalent to BP.
References


