On finite groups in which all minimal subgroups are $BNA$-subgroups

Yanhui Wang (*) – Xiuyun Guo (**)

Abstract – A subgroup $H$ of a group $G$ is said to be a $BNA$-subgroup of $G$ if either $H^x = H$ or $x \in \langle H, H^x \rangle$ for all $x \in G$. The purpose of this paper is first to give the best bound for the Fitting height of $G$ if all minimal subgroups of $G$ are $BNA$-subgroups of $G$, and next give an answer for the question of the paper [On $BNA$-normality and solvability of finite groups, Rend. Sem. Mat. Univ. Padova 136 (2016), 51-60]. Finally we use few $BNA$-subgroups of prime order to determine the structure of the finite groups. In fact, some new conditions for a finite group to be supersolvable have been given.

Mathematics Subject Classification (2020) – Primary 20F16; Secondary 20E34.

Keywords – $BNA$-subgroup, minimal subgroup, supersolvable group.

1. Introduction

There has been much interest to investigate the structure of finite groups under the assumption that minimal subgroups of a finite group $G$ have some kind of properties in $G$ [1–3, 11]. For example, Bellester-Bolinches and Guo [2] studied the class of finite groups for which every minimal subgroup is complemented. They prove that this class is just the class of all finite supersolvable groups with elementary abelian Sylow subgroups. Itô proved that if $p$ is an odd prime and all minimal subgroups of order $p$ of $G$ are contained in the center of the finite group $G$, then $G$ is $p$-nilpotent.
Later Buckley [3] proved that if $G$ is a finite group of odd order and all minimal subgroups of $G$ are normal in $G$, then $G$ is supersolvable. However, if $p = 2$ or $G$ is a group of even order, then the above corresponding results are all not true. Although many interesting results in this line have been given, there are still many issues that people are very interested in, such as, how about the bound for the 2-length if every minimal subgroup of a finite group is normal.

Now recall that a subgroup $H$ of a group $G$ is said to be abnormal if $x \in \langle H, H^x \rangle$ for all $x \in G$. The interesting thing is that a group $G$ is the only one subgroup such that it is both normal and abnormal in $G$ and that every maximal subgroup of $G$ is either normal or abnormal. So many authors investigated the structure of finite groups by using assumption that subgroups of a finite group are either normal or abnormal (for example, see [4, 5, 9]). Recently, the authors in [6] introduce a concept about subgroups—called $BNA$-subgroups. A subgroup $H$ of a finite group $G$ is called to be a $BNA$-subgroup of $G$ if either $H^x = H$ or $x \in \langle H, H^x \rangle$ for all $x \in G$. It is clear that both normal subgroups and abnormal subgroups are $BNA$-subgroups. Of course, there exist $BNA$-subgroups which are neither normal subgroups nor abnormal subgroups. For example, every cyclic subgroup of order 4 in $S_4$, the Symmetric group of degree 4, is a $BNA$-subgroup, but it is neither normal nor abnormal, which means that it is meaningful to investigate the structure of finite groups by the $BNA$-subgroups. In fact, the authors in [6, 7] studied the structure of finite groups by the assumptions that all cyclic subgroups of prime power order or all minimal subgroups are $BNA$-subgroups, and many interesting results have been given. We should mention the following results:

**Theorem 1.1.** [6, Theorem 3.3 (5)] Suppose that all minimal subgroups of a finite group $G$ are $BNA$-subgroups of $G$. Then the Fitting height of $G$ is bounded by 4.

Also the authors asked the following question in the end of the paper.

**Question 1.2.** [6, Question, P60] Are there finite groups $G$ such that every minimal subgroup of $G$ is a $BNA$-subgroup and $l_2(G) \geq 2$?

In the present paper we first prove that the Fitting height of $G$ is bounded by 3 if all minimal subgroups of $G$ are $BNA$-subgroups of $G$. Also we find a finite group $G$ such that the Fitting height of $G$ is just 3 and all minimal subgroups of $G$ are $BNA$-subgroups of $G$, which means that this bound 3 is best. Furthermore, this finite group $G$ satisfies $l_2(G) = 2$. So the above question has been answered. In the rest of the paper, we continue to investigate the structure of finite groups by using the minimal subgroups. However, we drop the assumption that every minimal subgroup is a $BNA$-subgroup of $G$. We want to use few $BNA$-subgroups of prime order to determine the structure of the finite groups. In fact, some new conditions for a finite group to be supersolvable have been given.
2. Preliminary results

In this section we collect some lemmas and some known concepts which will be frequently used in the sequel.

**Lemma 2.1.** [6, Lemma 2.1] Let $G$ be a finite group, $H \leq K \leq G$ and $N \leq G$. Suppose that $H$ is a BNA-subgroup of $G$. Then

1. $H$ is a BNA-subgroup of $K$.
2. $HN$ is a BNA-subgroup of $G$.
3. $HN/N$ is a BNA-subgroup of $G/N$.

**Lemma 2.2.** [6, Lemma 2.2 (2)] Let $H$ be a BNA-subgroup of a finite group $G$. If $H$ is subnormal in $G$, then $H$ is normal in $G$.

**Lemma 2.3.** [8, 9.1 Satz] If $G$ is a finite $p$-supersolvable group, then $G'$ is $p$-nilpotent. If $G$ is a finite supersolvable group, then $G'$ is nilpotent.

Let $p$ be a prime and $G$ a finite $p$-solvable group. Then the upper $p'$-$p$-series

$$1 = P_0 \leq N_0 < P_1 < N_1 < P_2 < \ldots < P_l \leq N_l = G$$

could be inductively defined by the rule that $N_k/P_k$ is the greatest normal $p'$-subgroup of $G/P_k$, and $P_{k+1}/N_k$ the greatest normal $p$-subgroup of $G/N_k$. The number $l$, which is the least integer such that $N_l = G$, is called the $p$-length of $G$, denoted by $l_p(G)$.

Recall that the product of all the normal $p$-nilpotent subgroups of a finite group $G$ is clearly $O_{p'}(G)$: that is the maximal normal $p$-nilpotent subgroups of $G$, which is called the $p$-Fitting subgroup of $G$ and denoted by $F_p(G)$.

Next we recall the concept of the $p$-Frattini subgroup. Set

$$S = \{M \text{ is a maximal subgroup in } G \mid [G : M] \text{ is a power of } p\}.$$

Then the $p$-Frattini subgroup of $G$, denoted by $\Phi_p(G)$, is defined as

$$\Phi_p(G) = \cap_{M \in S} M \text{ if } S \text{ is non empty}$$

and $\Phi_p(G) = G$ if $S$ is empty.

It is clear that $\Phi_p(G)$ is a characteristic subgroup of $G$ and the Frattini subgroup $\Phi(G)$ of $G$ is contained in $\Phi_p(G)$. It is also clear that $O_{p'}(G) \leq \Phi_p(G) \leq F_p(G)$ and $O_{p'}(G)$ is the Hall $p'$-subgroup of $\Phi_p(G)$ if $G$ is $p$-solvable. Furthermore, we may prove the following:

**Lemma 2.4.** Let $p$ be a prime and let $G$ be a finite $p$-solvable group. Then $\Phi_p(G)/O_{p'}(G) = \Phi(G/O_{p'}(G))$. 

Proof. It is clear that $\Phi(G/O_{p'}(G)) \leq \Phi_p(G)/O_{p'}(G)$. Conversely, let $M$ be a maximal subgroup of $G$ with $O_{p'}(G) \leq M$. If $\Phi_p(G) \not\leq M$, then the maximality of $M$ implies that $\Phi_p(G)M = G$. Thus $[G : M] = [\Phi_p(G) : \Phi_p(G) \cap M]$. Noticing that $O_{p'}(G) \leq M$ and $O_{p'}(G)$ is the Hall $p'$-subgroup of $\Phi_p(G)$, we see $[G : M]$ is a power of $p$ and therefore $\Phi_p(G) \leq M$, a contradiction. Hence $\Phi_p(G) \leq M$ for every maximal subgroup of $G$ with $O_{p'}(G) \leq M$, and therefore $\Phi_p(G)/O_{p'}(G) \leq \Phi(G/O_{p'}(G))$. The lemma is proved.

3. The Fitting height and the 2-length

In this section we discuss the Fitting height and the 2-length of finite groups in which every minimal subgroup is a $BNA$-subgroup.

Theorem 3.1. If all minimal subgroups of a finite group $G$ are $BNA$-subgroups of $G$, then the Fitting height of $G$ is bounded by 3.

Proof. By [6, Theorem 3.3(3)], $G$ is $p$-supersolvable for every odd prime $p$ dividing $|G|$ and therefore $G'$ is $p$-nilpotent by Lemma 2.3 for every odd prime $p$ in $\pi(G)$. Let $T_p$ be the normal $p$-complement of $G'$. Then

$$\bigcap_{p \neq 2} T_p$$

is the Sylow 2-subgroup of $G'$, denoted by $P$. It is clear that the Hall 2'-subgroups of $G'$ is nilpotent. It follows that

$$P \leq F_1(G), \quad G' \leq F_2(G), \quad G \leq F_3(G)$$

and so the Fitting height of $G$ is bounded by 3.

The following example illustrates that 3 is the best bound for the Fitting height of the above kind of finite groups, and it also gives an answer for the above question.

Example 3.2. Let $H = \langle c, d \mid c^9 = d^4 = 1, c^d = c^{-1} \rangle = \langle c \rangle \rtimes \langle d \rangle$. Then it is clear that $N = \langle c^3 \rangle \langle d^2 \rangle$ is normal in $H$ with $d^2 \in Z(H)$ and $\langle c^3 \rangle \leq H$, and that $H/N \cong S_3$. Also let $Q_8$ is an quaterian group of order 8. Since $H/N$ can be seen as a subgroup of $S_4$ and the automorphism group $\text{Aut}(Q_8)$ of $Q_8$ is isomorphic to $S_4$, the Symmetric group of degree 4, there exists an action from $H$ to $Q_8$ such that

$$C_H(Q_8) = \text{Ker}(H \text{ on } Q_8) = N.$$
Now let $G$ be the semidirect product $[Q_8]H$ of $Q_8$ and $H$ by using the above action. Then $P = Q_8 \langle d \rangle \in \text{Syl}_2(G)$ and $R = \langle c \rangle \in \text{Syl}_3(G)$. Furthermore, we may verify that every minimal subgroup of $P$ and $R$ is normal in $G$ and therefore every minimal subgroup of $G$ in normal. In this case, it is clear that

$$F_1(G) = Q_8 \langle c^3 \rangle \langle d^2 \rangle, \quad F_2(G) = Q_8 \langle c \rangle \langle d^2 \rangle, \quad F_3(G) = G.$$  

It is also clear that

$$O_{2'}(G) = \langle c^3 \rangle, \quad O_{2'2}(G) = Q_8 \langle c^3 \rangle \langle d^2 \rangle,$$

$$O_{2'22'}(G) = Q_8 \langle c \rangle \langle d^2 \rangle, \quad O_{2'22'2}(G) = G.$$ 

Thus the Fitting height of $G$ is 3 and the 2-length of $G$ is 2.

**Remark 3.3.** By the above discussion, we see that 3 is the best bound for the Fitting height of finite groups in which every minimal subgroup is normal.

### 4. New sufficient conditions for $p$-supersolvability

In this section we use few $BNA$-subgroups of prime order to determine the structure of the finite groups. In fact, some new conditions for a finite group to be supersolvable have been given.

**Lemma 4.1.** Let $p$ be a prime and $G$ a finite $p$-solvable group. If $H/K$ is a cyclic group for every $p$-chief factor $H/K$ of $G$ between $O_{p'}(G)$ and $F_p(G)$, then $G/C_G(F_p(G)/O_{p'}(G))$ is supersolvable.

**Proof.** Let $\overline{G} = G/O_{p'}(G)$. Then $O_p(\overline{G}) = F_p(G)/O_{p'}(G)$. By the hypotheses, we may assume that

$$\overline{1} = \overline{N_0} < \overline{N_1} < \ldots < \overline{N_t} = F_p(G)/O_{p'}(G)$$

is a part of $\overline{G}$-chief series containing in $F_p(G)/O_{p'}(G)$ with $\overline{N_i}/\overline{N_{i-1}}$ cyclic of order $p$. It is clear $t \geq 1$. If $t = 1$, then it follows from $|N_1| = p$ that $\overline{G}/C_{\overline{G}}(N_1)$ is cyclic and therefore $\overline{G}/C_{\overline{G}}(N_1)$ is supersolvable. Now assume $t > 1$. By induction on $t$ $\overline{G}/C_{\overline{G}}(N_{t-1})$ is supersolvable. Since $\overline{N_t}/\overline{N_1}$ is normal in $\overline{G}/\overline{N_1}$, we may use induction again for $\overline{G}/\overline{N_1}$ and we have $\overline{G}/C_{\overline{G}}(N_t/N_1)$ is supersolvable.

Set $C = C_{\overline{G}}(N_{t-1}) \cap C_{\overline{G}}(\overline{N_t}/\overline{N_1})$. Then $\overline{G}/C$ is supersolvable. Since $\overline{N_t}/\overline{N_{t-1}}$ is cyclic, there exists $n_1 \in \overline{N_t}$ such that $\overline{N_t} = \langle N_{t-1}, n_1 \rangle$. Then, for any $x \in C$, $ux = u$ for
any $u \in \overline{N}_{t-1}$, and there exists $k(x) \in \overline{N}_1$ such that $n_1^{x} = n_1 k(x)$. Thus, noticing that $C_G(\overline{N}_{t-1}) \leq C_G(\overline{N}_1)$, we see

$$n_1 k(xy) = n_1^{xy} = (n_1 k(x))^y = n_1^y k(x) = n_1 k(y) k(x)$$

if $y \in C$. The commutativity of $\overline{N}_1$ implies that $k(xy) = k(x)k(y)$. This means that

$$\mu : x^\mu = k(x)$$

is a homomorphism from $C$ to $\overline{N}_1$ and the kernel of $\mu$ is just $C_G(\overline{N}_1)$. Hence $C/C_G(\overline{N}_1)$ is cyclic, so $G/C_G(\overline{N}_1)$ is supersolvable and therefore $G/C_G(F_p(G)/O_{p'}(G))$ is supersolvable.

**Lemma 4.2.** Let $p$ be an odd prime and let $G$ be a finite $p$-solvable group. If every minimal subgroup of order $p$ in $F_p(G)/O_{p'}(G)$ is a BNA-subgroup of $G/O_{p'}(G)$. Then every $p$-chief factor of $G$ between $O_{p'}(G)$ and $F_p(G)$ is cyclic.

**Proof.** Let $A/O_{p'}(G)$ be a minimal subgroup of order $p$ in $F_p(G)/O_{p'}(G)$. It is clear that $A/O_{p'}(G)$ is subnormal in $G/O_{p'}(G)$. Then, by Lemma 2.2, $A/O_{p'}(G)$ is normal in $G/O_{p'}(G)$ and therefore $G/C_G(A/O_{p'}(G))$ is cyclic with exponent dividing $p - 1$. Now let $T$ be the subgroup of $G$ generated by $O_{p'}(G)$, $G'$ and all elements of the form $g^{p-1}$ in $G$. Thus $T \leq C_G(A/O_{p'}(G))$. It follows from [8, 5.12 Satz] that for every $p'$-element $x$ in $T$, $xO_{p'}(G)$ acts trivially on $F_p(G)/O_{p'}(G)$. Hence, $G/C_G(H/K)$ is abelian with exponent dividing $p - 1$ for every $p'$-chief factor $H/K$ of $G$ between $O_{p'}(G)$ and $F_p(G)$, and therefore, by [13, Lemma 1.3], $H/K$ is cyclic for every $p'$-chief factor $H/K$ of $G$ between $O_{p'}(G)$ and $F_p(G)$.

**Theorem 4.3.** Let $p$ be an odd prime and $G$ a finite $p$-solvable group. If every minimal subgroup of order $p$ in $F_p(G)/O_{p'}(G)$ is a BNA-subgroup of $G/O_{p'}(G)$. Then $G$ is $p$-supersolvable.

**Proof.** Since $G$ is $p$-solvable, $C_G(F_p(G)/O_{p'}(G)) \leq F_p(G)$ by [10, 9.3.1]. Then Lemma 4.2 and Lemma 4.1 imply that $G/C_G(F_p(G)/O_{p'}(G))$ is supersolvable and so $G/F_p(G)$ is supersolvable. Clearly $(G/O_{p'}(G))/(F_p(G)/O_{p'}(G)) \cong G/F_p(G)$ and $F_p(G)/O_{p'}(G)$ is a $p$-group. By the hypothesis, every minimal subgroup of $F_p(G)/O_{p'}(G)$ is a BNA-subgroup of $G/O_{p'}(G)$, and so every minimal subgroup of $F_p(G)/O_{p'}(G)$ is in normal in $G/O_{p'}(G)$ by Lemma 2.2, it follows from Corollary 3 of [12] that $G/O_{p'}(G)$ is supersolvable. Therefore $G$ is $p$-supersolvable.

**Corollary 4.4.** Let $p$ be an odd prime and $G$ a finite $p$-solvable group. If every minimal subgroup of order $p$ in $F_p(G)$ is a BNA-subgroup of $G$. Then $G$ is $p$-supersolvable.
By using the arguments used in the proofs of the Lemmas 4.1 and 4.2, we may prove the following results.

**Lemma 4.5.** Let $p$ be a prime and $G$ a finite $p$-solvable group. If $H/K$ is cyclic for every $p$-chief factor $H/K$ of $G$ between $\Phi_p(G)$ and $F_p(G)$, then $G/C_G(F_p(G)/\Phi_p(G))$ is supersolvable.

**Lemma 4.6.** Let $p$ be an odd prime and let $G$ be a finite $p$-solvable group. If every minimal subgroup of order $p$ in $F_p(G)/\Phi_p(G)$ is a $BNA$-subgroup of $G/\Phi_p(G)$, then every $p$-chief factor of $G$ between $\Phi_p(G)$ and $F_p(G)$ is cyclic.

**Theorem 4.7.** Let $p$ be an odd prime and $G$ a finite $p$-solvable group. If every minimal subgroup of order $p$ in $F_p(G)/\Phi_p(G)$ is a $BNA$-subgroup of $G/\Phi_p(G)$, then $G$ is $p$-supersolvable.

**Proof.** It is clear that $O_{p'}(G/\Phi_p(G)) = 1$ and $O_p(G/\Phi_p(G)) = F_p(G)/\Phi_p(G) = F_p(G/\Phi_p(G))$. Since $G$ is $p$-solvable, we have that $C_{G/\Phi_p(G)}(F_p(G/\Phi_p(G))) \leq F_p(G/\Phi_p(G)) = F_p(G)/\Phi_p(G)$ by [10, 9.3.1]. Clearly $C_{G/\Phi_p(G)}(F_p(G/\Phi_p(G))) = C_{G/\Phi_p(G)}(F_p(G)/\Phi_p(G)) = C_G(F_p(G)/\Phi_p(G))/\Phi_p(G)$. It follows from Lemma 4.5 and Lemma 4.6 that $G/C_G(F_p(G)/\Phi_p(G))$ is supersolvable and so $G/F_p(G)$ is supersolvable. Clearly, $(G/\Phi_p(G))/(F_p(G)/\Phi_p(G)) \cong G/F_p(G)$ and $F_p(G)/\Phi_p(G)$ is a $p$-group. By the hypothesis, every minimal subgroup of $F_p(G)/\Phi_p(G)$ is a $BNA$-subgroup of $G/\Phi_p(G)$, and so every minimal subgroup of $F_p(G)/\Phi_p(G)$ is in normal in $G/\Phi_p(G)$ by Lemma 2.2, it follows from Corollary 3 of [12] that $G/\Phi_p(G)$ is supersolvable. Since $(G/O_{p'}(G))/(\Phi_p(G)/O_{p'}(G)) = (G/O_{p'}(G))/\Phi(G/O_{p'}(G))$, $G/O_{p'}(G)$ is supersolvable. Therefore $G$ is $p$-supersolvable. 

**Acknowledgements** – The authors would like to thank the referee for his/her valuable suggestions and useful comments contributed to the version of this paper.

**Funding** – This work was partially supported by the National Natural Science Foundation of China (12171302, 11501331).

**References**


