ON HYPERCENTRE-BY-POLYCYCLIC-BY-NILPOTENT GROUPS

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ABSTRACT. If $\{\gamma^{s+1}G\}$ and $\{\zeta_s(G)\}$ denote respectively the lower and upper central series of the group G, $s \geq 0$ an integer, and if $\gamma^{s+1}G/(\gamma^{s+1}G \cap \zeta_s(G))$ is polycyclic (resp. polycyclic-by-finite) for some s, then we prove that $G/\zeta_{2s}(G)$ is polycyclic (resp. polycyclic-by-finite). The corresponding result with polycyclic replaced by finite was proved in 2009 by G. A. Fernández-Alcober and M. Morigi. We also present an alternative approach to the latter.

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If G is any group let $\{\gamma^{s+1}G\}$ denote the lower central series of G and $\{\zeta_s(G)\}$ the upper central series of G; throughout $s \geq 0$ and $m \geq 1$ denote integers. In their very nice paper [2] Fernández-Alcober and Morigi prove the following very interesting result. If G is a group with the index $(\gamma^{s+1}G : (\gamma^{s+1}G) \cap \zeta_s(G)) = m$, then $(G : \zeta_{2s}(G))$ is finite and their proof yields that it is bounded by a function of m and s only (although no specific such function is recorded in [2]; see Propositions 1 and 2 below of a bound). Also see [2] for the history of this theorem. We prove here the following variant.

THEOREM. Let G be a group and $s \geq 0$ an integer with $\gamma^{s+1}G/(\gamma^{s+1}G \cap \zeta_s(G))$ polycyclic-by-finite. Then $G/\zeta_{2s}(G)$ is polycyclic-by-finite.

If in the theorem $\gamma^{s+1}G/(\gamma^{s+1}G\cap\zeta_s(G))$ is polycyclic, then G is soluble. Thus the following corollary is immediate.

COROLLARY. Let G be a group and $s \geq 0$ an integer with $\gamma^{s+1}G/(\gamma^{s+1}G\cap\zeta_s(G))$ polycyclic. Then $G/\zeta_{2s}(G)$ is polycyclic.

We cannot replace polycyclic-by-finite in the theorem by Chernikov, see Example 4 of [8]; for there γ^2G is Chernikov, $\zeta_1(G) = \langle 1\rangle$ and G is not even periodic.

One particular, indeed the original, special case of Fernández-Alcober and Morigi's theorem is P. Hall's Theorem 2 in [3]. He proved that if $|\gamma^{s+1}G|$ divides m, then $(G : \zeta_{2s}(G))$ finite and in fact divides m if s = 0 and is at most m to the power of $\mathbf{e}(m)^s(\log_2 m)^s+\mathbf{e}(m)$ in general, where if $m = \Pi_p\, p^{e(p)}$, for p running over all primes, then $\mathbf{e}(m) = \Sigma_p\, e(p) \leq \log_2 m$. Baer's analogue of Hall's theorem (e.g. see [6] 14.5.1) is that if $(G : \zeta_s(G))$ divides m, then $|\gamma^{s+1}G|$ is finite and divides a bounded power of m (e.g. see [8] for discussion of the bounds here and above). However, in the situation of Hall's Theorem, for very simple reasons, there is no need for $(G : \zeta_{2s}(G))$ to divide any power of m. (If G = Sym(3), s = 1 and m = 3, then $(G : \zeta_{2s}(G)) = 6$.) However, we will see below that not all is lost.

For each prime p define the integer-valued function $f_p(s, m)$ for each pair of integers s and m by $f_p(s, m)$ is the least integer such that if G is a finite p-group with $(\gamma^{s+1}G : \gamma^{s+1}G\cap\zeta_s(G))$ dividing m, then $(G : \zeta_{2s}(G))$ divides $f_p(s, m)$. Clearly $f_p(s, m)$ exists by Fernández-Alcober and Morigi's theorem and is a power of p; further if p > m, or more generally if p is prime to m, then $\gamma^{s+1}G \leq \zeta_s(G)$, $G = \zeta_{2s}(G)$ and $f_p(s, m) = 1$. Set $f(s, m) = \Pi_p\, f_p(s, m)$. Then f(s, m) for each s and m is a well-defined integer dividing a power of m. A special case of Casolo, Dardano and Rinauro's Theorem A in [1] is that if $L = \gamma^{s+1}G$ is finite, then the index $(G : \zeta(G))$ is finite and divides (see proof in [1] or comments on Theorem F in [8]) $|\mathrm{Aut}L||\zeta_1(L)|$; here $\zeta(G)$ denotes $\bigcup_s \zeta_s(G)$, the hypercentre of G (not as with some authors the centre of G). Let h(m) denote the least integer such that if $|\gamma^{s+1}G|$ divides m for some s, then $(G : \zeta(G))$ divides h(m). The above implies that h(m) divides m.m!.

PROPOSITION 1. Let G be a group with $(\gamma^{s+1}G : \gamma^{s+1}G\cap\zeta_s(G))$ dividing m. Then $(\zeta(G) : \zeta_{2s}(G))$ divides f(s, m), $(G : \zeta(G))$ divides h(m) which divides m.m! and $(G : \zeta_{2s}(G))$ divides f(s, m).h(m).

Thus the full divisor of $(G : \zeta_{2s}(G))$ prime to m divides (m–1)! and as such is likely to be much smaller than the full divisor of $(G : \zeta_{2s}(G))$ dividing a power of m.

For all s, m and p as above we can bound $f_p(s, m)$ as follows. Let p^e be the largest power of p to divide m. If e = 0 then $f_p(s, m) = 1$, so assume $e \geq 1$. Also $f_p(0, m) = p^e$, so assume $s \geq 1$.

Put k(0) = e and define k(i) and l(i) inductively by setting

$$k(i) = k(i-1)^2(k(i-1).(s+i))^{k(i-1)+s+i} \text{ and}$$

$$l(i) = k(i-1)(k(i-1) - 1)/2.$$

Then set $h(s, m) = \Sigma_{1\leq j\leq s} (k(j-1)k(j)+l(j))$.

PROPOSITION 2. Then $\log_p f_p(s, m) \leq e.h(s, m)^s+h(s, m)$; that is, $f_p(s, m)$ divides p to the power of $e.h(s, m)^s + h(s, m)$.

This bound for $f_p(s, m)$ is certainly too large. For a start there are various places in our proof of Proposition 2 where we have deliberately used an unnecessarily large estimate in order to prevent the calculations getting too unwieldy.

THE PROOFS

Below, just for the specific group denoted by G, Γ^{i+1} denotes $\gamma^{i+1}G$ and Z_i denotes $\zeta_i(G)$. Also **P** denotes the class of polycyclic groups and **PF** the class of polycyclic-by-finite groups. To prove the theorem we follow the general strategy of [2].

LEMMA 1. Let $X \geq X_1 \geq X_2 \geq \ldots \geq X_r = \langle 1\rangle$ be a normal series of finite length of the group X with $Y \leq \mathrm{Aut}X$ the stability group of this series. If X/X_1 is d-generator (d finite) and if $X_1 \in$ **PF**, then $Y \in$ **PF**.

Proof. If r = 1 then $Y = \langle 1\rangle$, so assume $r \geq 2$. By stability theory the factor $Y/C_Y(X/X_2)$ embeds into the direct product of d copies of X_1/X_2 (e.g. [4] 1.C.3) and hence is **PF**. Now X/X_2 is finitely generated. Also $C_Y(X/X_2)$ stabilizes the series $X \geq X_2 \geq \ldots \geq X_r = \langle 1\rangle$ and hence by induction on r is **PF**. Consequently $Y \in$ **PF**.

LEMMA 2. Let X and Y be subgroups of a group G such that X is finitely generated, X normalizes Y and $[X, Y] \in$ **PF**. Suppose also that $[X, Y] \leq \zeta_t(Y)$ for some finite t. Then $C = C_Y(X[X, Y])$ is normal in XY and $Y/C \in$ **PF**.

This lemma replaces Lemma 2.1 of [2].

Proof. Apply Lemma 1 with X[X, Y] for X and Y/C for Y.

LEMMA 3. Let H, K, M and N be normal subgroups of a group G with $M \leq H$, $N \leq K$, $H/M \in$ **PF**, $K/N \in$ **PF** and $[H, N] = \langle 1 \rangle = [K, M]$. Then $[H, K] \in$ **PF**.

This is a special case of [5] 4.22 (note that **PF** does satisfy the hypotheses of [5] 4.22, see [5] Page 115). Lemma 3 replaces Theorem 2.2 of [2]. Lemma 4 below is the critical part of our proof of the theorem. It is the analogue of the important Proposition 2.3 of [2] and our proof of Lemma 4 closely follows the strategy of the proof of the latter in [2]. If G is also nilpotent there is an easy proof of Lemma 4 using Lemma 8 below but I do not see how one can use that approach to prove Lemma 4 in general.

LEMMA 4. Let s be a positive integer and G a group with $\Gamma^s/Z \in$ **PF** for $Z = \Gamma^s \cap Z_1$. Then $G/C_G(\Gamma^s) \in$ **PF** and $\Gamma^{s+1} \in$ **PF**.

Proof. Note first that G is soluble-by-finite. Also if $G/C_G(\Gamma^s) \in$ **PF**, then $\Gamma^{s+1} \in$ **PF** by Lemma 3 applied with Γ^s, G, Z and $C_G(\Gamma^s)$ for H, K, M and N. Thus we focus on the first claim of the lemma. Set $C = C_G(\Gamma^s/Z)$ and note that C is nilpotent (of class at most s+1). Also $G/C \in$ **PF** by a theorem of (independently) Smirnov and Baer, see [7] 5.2.

There exists a finitely generated subgroup U of G with G = UC and $\Gamma^s = (\gamma^s U)Z$. Now if X is any finitely generated subgroup of G, then $X/(X\cap\Gamma^s)$ is nilpotent and hence $X/(X\cap Z) \in$ **PF**. Then X satisfies the maximal condition on normal subgroups (e.g. [7] 3.10) and $X\cap Z$ is central. Thus $X\cap Z$ is finitely generated and $X \in$ **PF**. In particular every section of U is finitely generated (and U can be chosen with its minimal number of generators bounded in terms of s and the isomorphism class of Γ^s/Z if we wish).

We prove by reverse induction on j that for j = 1, 2, … , s there is a subgroup H_j of $\Gamma^j \cap C$ normal in $U(\Gamma^j \cap C)$ and such that $(\Gamma^j \cap C)/H_j \in \mathbf{PF}$ and $[H_j, \gamma^{s-j+1}U] = \langle 1 \rangle$. Clearly we may choose $H_s = Z$. Once we have completed this construction of the H_j we will have H_1 normal in $UC = G$, $C/H_1 \in \mathbf{PF}$ and $[H_1, \Gamma^s] = [H_1, (\gamma^s U)Z] = [H_1, \gamma^s U] = \langle 1 \rangle$. Since $G/C \in \mathbf{PF}$, we will have that $G/H_1 \in \mathbf{PF}$ and the proof of the lemma will be complete.

Suppose we have constructed H_{j+1} for some $j \geq 1$ with H_{j+1} normal in $U(\Gamma^{j+1} \cap C)$ with $(\Gamma^{j+1} \cap C)/H_{j+1} \in \mathbf{PF}$ and $[H_{j+1}, \gamma^{s-j}U] = \langle 1 \rangle$. Set

$$K_j = \Gamma^j \cap C_C((\gamma^{s-j}U)[\gamma^{s-j}U, \Gamma^j \cap C]Z/Z).$$

Clearly U normalizes K_j. Now $\gamma^{s-j}U$ is finitely generated and $[\gamma^{s-j}U, \Gamma^j]Z/Z \leq \Gamma^s/Z \in \mathbf{PF}$. Also C is nilpotent. Therefore K_j is normal in $U(\Gamma^j \cap C)$ and $(\Gamma^j \cap C)/K_j \in \mathbf{PF}$, this by Lemma 2 (where modulo Z, X is $\gamma^{s-j}U$ and Y is $\Gamma^j \cap C$). Further

(a) … $[K_j, \gamma^{s-j}U, U] \leq [Z, U] = \langle 1 \rangle$.

Consider $D_{j+1} = \Gamma^{j+1} \cap C_C(\gamma^{s-j}U)$. Clearly U normalizes D_{j+1}. Also

$$[\Gamma^j, \gamma^{s-j}U, D_{j+1}] \leq [\Gamma^s, D_{j+1}] = [(\gamma^s U)Z, D_{j+1}] = \langle 1 \rangle$$

since $\gamma^s U \leq \gamma^{s-j}U$. Further $[\gamma^{s-j}U, D_{j+1}, \Gamma^j] = \langle 1 \rangle$ by the definition of D_{j+1}. By the three subgroup lemma we have $[D_{j+1}, \Gamma^j, \gamma^{s-j}U] = \langle 1 \rangle$. Consequently D_{j+1} is normal in $T_j = U\Gamma^j$. By hypothesis $[H_{j+1}, \gamma^{s-j}U] = \langle 1 \rangle$, where $H_{j+1} \leq \Gamma^{j+1} \cap C$. Therefore $H_{j+1} \leq D_{j+1}$ and hence $(\Gamma^{j+1} \cap C)/D_{j+1} \in \mathbf{PF}$.

We now work in T_j/D_{j+1}. Since $[\Gamma^j \cap C, U]D_{j+1}/D_{j+1} \leq (\Gamma^{j+1} \cap C)/D_{j+1}$, so $[\Gamma^j \cap C, U]D_{j+1}/D_{j+1} \in \mathbf{PF}$. Set $L_j = \Gamma^j \cap C_C(U[U, \Gamma^j \cap C]D_{j+1}/D_{j+1})$. Clearly U normalizes L_j and Lemma 2 yields that L_j is normal in $\Gamma^j \cap C$ with $(\Gamma^j \cap C)/L_j \in \mathbf{PF}$. Also

(b) … $[U, L_j, \gamma^{s-j}U] \leq [D_{j+1}, \gamma^{s-j}U] = \langle 1 \rangle$.

Set $H_j = K_j \cap L_j$. Then $(\Gamma^j \cap C)/H_j \in \mathbf{PF}$. Also H_j is normal in $U(\Gamma^j \cap C)$ and (a) and (b) and the three subgroup lemma yield that $[\gamma^{s-j}U, U, H_j] = \langle 1 \rangle$. Thus $[\gamma^{s-j+1}U, H_j] = \langle 1 \rangle$. The construction of H_j and hence the proof of the lemma are now complete.

LEMMA 5. Let s and t be positive integers and G a group with $\Gamma^s/(\Gamma^s \cap Z_t) \in \mathbf{PF}$. Then $G/C_G(\Gamma^{s+j}/(\Gamma^{s+j} \cap Z_{t-j-1}))$ and $\Gamma^{s+j+1}/(\Gamma^{s+j+1} \cap Z_{t-j-1})$ are both in $\mathbf{PF}$ for $0 \leq j < t$.

Proof. Lemma 4 applied to G/Z_{t-1} yields that $G/C_G(\Gamma^s/(\Gamma^s \cap Z_{t-1})) \in$ **PF** and $\Gamma^{s+1}/(\Gamma^{s+1} \cap Z_{t-1}) \in$ **PF**. Now apply induction on t to Γ^{s+1}.

LEMMA 6 (see [2] Lemma 2.5). For any group G and positive integer s set $H = \bigcap_{1\leq j\leq s} C_G(\Gamma^{s+j}/(\Gamma^{s+j} \cap Z_{s-j}))$. Then $[H, {}_{t-1}G, H] \leq Z_{2s-t}$ for $1 \leq t \leq 2s$.

Thus changing the notation slightly in Lemma 6 we have $[H, {}_{s-u}G, H] \leq Z_{s+u-1}$ for $0 \leq u \leq s$. Hall's proof of his theorem, specifically the proof of 14.5.3 in [6], yields the following.

LEMMA 7 (effectively P. Hall). Let H be a normal subgroup of a group G with G/H (finitely) d-generated. Suppose $[H, {}_{s-u}G, H] \leq Z_{s+u}$ for $0 \leq u \leq s$. Then $H/(H \cap Z_{2s})$ is isomorphic to a section of the direct product of d^s copies of $[H, {}_sG]/([H, {}_sG] \cap Z_s)$ and hence to a section of the direct product of d^s copies of $\Gamma^{s+1}/(\Gamma^{s+1} \cap Z_s)$.

The Proof of the Theorem. Set $H = \bigcap_{1\leq j\leq s} C_G(\Gamma^{s+j}/\Gamma^{s+j} \cap Z_{s-j})$. By Lemma 5 we have that $G/H \in$ **PF**. Also H satisfies by Lemma 6 the hypotheses of Lemma 7. Since here $\Gamma^{s+1}/(\Gamma^{s+1} \cap Z_s) \in$ **PF**, so $H/(H \cap Z_{2s}) \in$ **PF**. Consequently $G/(H \cap Z_{2s}) \in$ **PF** and $G/Z_{2s} \in$ **PF**.

Can we replace **PF** in the above proofs by some other class **X** of groups? To keep basically to the proofs above we would need **X** to satisfy at least the conditions on **X** required for [5] 4.22 and also to satisfy the hypotheses labelled ii) on Page 119 of [5]. I have no idea whether any of these eight conditions would prove redundant. Also apart from **P** and **PF** I know of no interesting classes containing some infinite groups and satisfying all these eight conditions.

The Proof of Proposition 1. Again set $\Gamma^i = \gamma^i G$ and $Z_i = \zeta_i(G)$ for each i. By definition of h(m) the index $(G : \zeta(G)) = (G/Z_s : \zeta(G/Z_s))$ divides h(m). Suppose first that G is finite. Let $\zeta(G) = \times_p Q_p$, where Q_p is a p-group and choose a Sylow p-subgroup P_p of G (necessarily) containing Q_p. By stability theory (e.g. by [7] 1.21c) applied to $\{Q_p \cap Z_i\}_{i\geq 0}$) the quotient $G/C_G(Q_p)$ is a p-group, so $G = C_G(Q_p)P_p$ and hence $[\zeta_{i+1}(P_p) \cap Q_p, G] \leq \zeta_i(P_p) \cap Q_p$. Consequently $[\zeta_{2s}(P_p) \cap Q_p, {}_{2s}G] = \langle 1 \rangle$ and $\zeta_{2s}(P_p) \cap Q_p =$

$Z_{2s} \cap Q_p$. By definition of $f_p(s, m)$, $(P_p : \zeta_{2s}(P_p))$ divides $f_p(s, m)$. Therefore $(Q_p : Z_{2s} \cap Q_p)$ divides $f_p(s, m)$ and hence $(\zeta(G) : Z_{2s})$ divides $f(s, m)$. Consequently $(G : \zeta_{2s}(G))$ divides $f(s, m).h(m)$.

Now assume that G is just finitely generated. Since by hypothesis G/ Z_s is finite-by-nilpotent, G is nilpotent-by-finite and hence polycyclic-by-finite (e.g. [7] 2.13). Choose a prime $p > f(s, m).h(m)$. There exists a normal subgroup N of G of finite index with N residually a finite p-group (e.g. [7] 4.8 and 4.10 or use 2.16). Set $R_r = N^r$ for $r = p^e$. Then G/R_r is finite. By the finite case there exists a normal subgroup $S_r \geq R_r$ of G with $[S_r, {}_{2s}G] \leq R_r$, with $(\zeta(G)S_r : S_r)$ dividing $f(s, m)$ and with $(G : S_r)$ dividing $f(s, m).h(m)$. Since $p > f(s, m).h(m)$, so $N \leq S_r$. But G/N has only a finite number of subgroups, so there exists an infinite set X of integers $r = p^e$ with $S_r = S$ for all r in X. Clearly $\bigcap_X R_r = \langle 1 \rangle$. Therefore $[S, {}_{2s}G] = \langle 1 \rangle$, $S \leq Z_{2s} \leq \zeta(G)$, $(\zeta(G) : S)$ divides $f(s, m)$ and $(G : S)$ divides $f(s, m).h(m)$. This completes the finitely generated case.

Finally we consider the general case. Thus now we are only assuming that $(\Gamma^{s+1} : \Gamma^{s+1} \cap Z_s)$ divides m. Consider finitely generated subgroups $X_0 \leq X \leq Y$ of G. Then $\gamma^{s+1}X \leq \Gamma^{s+1} \cap X$, $\zeta_s(X) \geq X \cap Z_s$ and $(\gamma^{s+1}X : \gamma^{s+1}X \cap \zeta_s(X))$ divides m. Consequently $(\zeta(X) : \zeta_{2s}(X))$ divides $f(s, m)$ and $(X : \zeta_{2s}(X))$ divides $f(s, m).h(m)$.

Firstly choose X_0 so that $(X_0 : \zeta_{2s}(X_0)$ is maximal and secondly among such X_0 choose X_0 so that $(X_0 : \zeta(X_0))$ is maximal. Now

$$(X_0 : \zeta_{2s}(X_0)) \leq (X_0 : X_0 \cap \zeta_{2s}(X)) = (X_0\zeta_{2s}(X) : \zeta_{2s}(X)) \leq (X : \zeta_{2s}(X)).$$

By the choice of X_0 these inequalities are equalities. In particular $X_0 \cap \zeta_{2s}(X) = \zeta_{2s}(X_0)$, $X_0\zeta_{2s}(X) = X$ and $(X : \zeta_{2s}(X)) = (X_0 : \zeta_{2s}(X_0))$. Arguing with $X \leq Y$ in place of $X_0 \leq X$ we have $X \cap \zeta_{2s}(Y) = \zeta_{2s}(X)$ and hence $\zeta(X) \cap \zeta_{2s}(Y) = \zeta_{2s}(X)$. It follows that $Z_* = \bigcup_X \zeta_{2s}(X)$ is a normal subgroup of G with $[Z_*, {}_{2s}G] = \langle 1 \rangle$. Clearly $Z_{2s} \leq \bigcup_X X \cap Z_{2s} \leq \bigcup_X \zeta_{2s}(X)$, so $Z_{2s} = Z_*$. Also $X_0Z_{2s} = \bigcup_X X_0\zeta_{2s}(X) = \bigcup_X X = G$ and $X_0 \cap Z_{2s} \geq \zeta_{2s}(X_0)$. Hence $(G : Z_{2s})$ divides $(X_0 : \zeta_{2s}(X_0))$, which divides $f(s, m).h(m)$.

Set $n = 2s + f(s, m).h(m)$. Then $\zeta(G) = Z_n$ and $\zeta(X) = \zeta_n(X)$ for each X. Thus arguing as above with n replacing 2s and using the maximal choice of $(X_0 : \zeta(X_0))$ we have that $\zeta(G) = \bigcup_X \zeta(X)$. Also $(\zeta(G) : Z_{2s})$ is finite, so there exists X with $\zeta(G) = \zeta(X)Z_{2s}$. Further $\zeta(X) \cap Z_{2s} = \bigcup_Y (\zeta(X) \cap \zeta_{2s}(Y)) = \zeta_{2s}(X)$. Therefore $\zeta(G)/Z_{2s} \cong$

$\zeta(X)/\zeta_{2s}(X)$ and consequently $(\zeta(G) : Z_{2s})$ divides f(s, m). The proof of the proposition is complete.

LEMMA 8. Let G be a group, U a subgroup of G and $s \geq 0$ an integer with $\Gamma^{s+1} \leq (\gamma^{s+1}U)Z_1$. If either $\gamma^{s+2}U$ is normal in G or G is nilpotent, Then $\Gamma^{s+2} = \gamma^{s+2}U$.

Proof. If s = 0, then $G \leq UZ_1$ and $\Gamma^2 = \gamma^2U$. Thus the claim holds for s = 0. Assume $s \geq 1$ and consider first the case where $\gamma^{s+2}U$ is normal in G. We prove by induction on j that

(*) $\quad [\Gamma^{s+2-j}, \gamma^jU] \leq \gamma^{s+2}U$

whenever $1 \leq j \leq s+1$. If j = 1 then $[\Gamma^{s+1}, U] \leq [(\gamma^{s+1}U)Z_1, U] = \gamma^{s+2}U$. Thus (*) holds for j = 1. Note that if (*) holds for j = s+1, then $[G, \gamma^{s+1}U] \leq \gamma^{s+2}U$. Consequently

$$\Gamma^{s+2} = [\Gamma^{s+1}, G] \leq [(\gamma^{s+1}U)Z_1, G] = [\gamma^{s+1}U, G] \leq \gamma^{s+2}U$$

and we will have proved that $\Gamma^{s+2} = \gamma^{s+2}U$ in this case.

Suppose (*) holds for some j with $1 \leq j \leq s$. Clearly

$$[\Gamma^{s+2-(j+1)}, \gamma^{j+1}U] = [\gamma^jU, U, \Gamma^{s+1-j}].$$

Now

$$[\Gamma^{s+1-j}, \gamma^jU, U] \leq [\Gamma^{s+1}, U] \leq [(\gamma^{s+1}U)Z_1, U] = \gamma^{s+2}U.$$

Also

$$[U, \Gamma^{s+1-j}, \gamma^jU] \leq [\Gamma^{s+2-j}, \gamma^jU] \leq \gamma^{s+2}U \text{ by (*) for j.}$$

But $\gamma^{s+2}U$ here is normal in G, so by the three subgroups lemma $[\gamma^jU, U, \Gamma^{s+1-j}] \leq \gamma^{s+2}U$. Therefore (*) holds for j+1. This completes the proof of (*).

Now consider the case where G is nilpotent. Trivially if $\gamma^{s+2}U$ is central in G it is normal in G. Hence by the above case applied to G/Γ^{s+3} we obtain $\Gamma^{s+2} = (\gamma^{s+2}U)\Gamma^{s+3}$. Applying it to G/Γ^{s+4} yields that $\Gamma^{s+3} = (\gamma^{s+3}U)\Gamma^{s+4}$ and hence that $\Gamma^{s+2} = (\gamma^{s+2}U)\Gamma^{s+4}$. A simple induction yields that $\Gamma^{s+2} = (\gamma^{s+2}U)\Gamma^t$ for all $t \geq s+3$. But G here is nilpotent. Therefore $\Gamma^{s+2} = \gamma^{s+2}U$.

LEMMA 9. Let G be a finite p-group (p a prime) on f generators. Suppose $s \geq 1$ and $e \geq 0$ are integers with $(\Gamma^s : \Gamma^s \cap Z_1) = p^e$. Then $\log_p|\Gamma^{s+1}| \leq e^2f^{s+e}$.

Proof. The minimal number of generators of Γ^j/Γ^{j+1} is bounded by Witt's function $w(j, f) = j^{-1}\Sigma_{d|j}\, \mu(d)f^{j/d} \leq f^j$. If $x \in \Gamma^s$ and $g \in G$, then $\phi_g : x \mapsto [x, g]$ determines a homomorphism of $\Gamma^s/(\Gamma^s \cap Z_1)$ into $\Gamma^{s+1}/\Gamma^{s+2}$ and Γ^{s+1} is generated by the images of the ϕ_g. Therefore $\Gamma^{s+1}/\Gamma^{s+2}$ has exponent dividing p^e and consequently so does Γ^j/Γ^{j+1} for all $j > s$. Clearly $\Gamma^{s+e} \leq Z_1$, so $\Gamma^{s+e+1} = <1>$ and hence

$$\log_p|\Gamma^{s+1}| \leq e.\Sigma_{s<j\leq s+e}\, w(j, f) \leq e(f^{s+1} + f^{s+2} + \ldots + f^{s+e}) \leq e^2 f^{e+s}.$$

Lemma 9 is proved.

LEMMA 10. Let G be a finite p-group and $s \geq 1$ and $e \geq 0$ integers with $(\Gamma^s : \Gamma^s \cap Z_1) = p^e$. Set $k = e^2(es)^{e+s}$ and $l = e(e - 1)/2$. Then $|\Gamma^{s+1}|$ divides p^k and $(G : C_G(\Gamma^s))$ divides $p^{ek + l}$.

Proof. Now Γ^s contains and is generated by the left-normed commutators of length s. Clearly Γ^s modulo Z_1 is generated by at most e of these. Therefore there is an es-generator subgroup U of G with $\Gamma^s \leq (\gamma^s U)Z_1$. By Lemma 8 we have $\Gamma^{s+1} = \gamma^{s+1}U$. Then Lemma 9 implies that $|\Gamma^{s+1}|$ divides p^k.

Set $C = C_G(\Gamma^s)$ and $D = C_G(\Gamma^s/(\Gamma^s \cap Z_1))$. Since G is a finite p-group, G/D stabilizes a central series of $\Gamma^s/(\Gamma^s \cap Z_1)$ of length e with its factors of order p. Therefore $|G/D|$ divides p^l. Also D stabilizes the series $\Gamma^s \geq \Gamma^s \cap Z_1 \geq <1>$. Hence by stability theory (e.g. [4] 1.C.3) the group D/C embeds into $\mathrm{Hom}(\Gamma^s/(\Gamma^s \cap Z_1), \Gamma^{s+1} \cap Z_1)$ and $\Gamma^s/(\Gamma^s \cap Z_1)$ is at most e-generator. Consequently D/C has order dividing $|\Gamma^{s+1}|^e$ and therefore $(G : C)$ divides p^{ek+l}.

The Proof of Proposition 2. Let G be a finite p-group and $s \geq 0$ and $e \geq 0$ integers with $(\Gamma^{s+1} : \Gamma^{s+1} \cap Z_s) = p^e$. (Note that $k(j)$, $l(j)$ and $h(s, m)$ are all increasing with e, so there is no harm in assuming this index equals p^e rather than divides p^e.) If e or s is zero the claim is trivial, so assume otherwise.

By Lemma 10 applied to G/Z_{s-1} we have

$$\log_p(\Gamma^{s+2} : \Gamma^{s+2} \cap Z_{s-1}) \leq e^2(e(s+1))^{e+s+1} = k(1)$$

$$\text{and } \log_p(G : C_G(\Gamma^{s+1}/(\Gamma^{s+1} \cap Z_{s-1})) \leq e.k(1) + l(1).$$

The same lemma but now applied to Γ^{s+2} and G/Z_{s-2} yields that

$$\log_p(\Gamma^{s+3} : \Gamma^{s+3} \cap Z_{s-2}) \leq k(2) \text{ and } \log_p(G : C_G(\Gamma^{s+2}/(\Gamma^{s+2} \cap Z_{s-2})) \leq k(1)k(2) + l(2).$$

We keep going in this way. Hence $\log_p(G : C_G(\Gamma^{s+j}/\Gamma^{s+j} \cap Z_{s-j})) \leq k(j-1)k(j) + l(j)$ for each $j \geq 1$.

Set $H = \bigcap_{1\leq j\leq s} C_G(\Gamma^{s+j}/(\Gamma^{s+j} \cap Z_{s-j}))$. Then

$$\log_p(G : H) \leq \Sigma_{1\leq j\leq s} (k(j-1)k(j) + l(j)) = h \text{ say}.$$

In particular G/H is generated by at most h elements. By Lemmas 6 and 7 the group $H/(H\cap Z_{2s})$ embeds into the direct product of h^s copies of $\Gamma^{s+1}/(\Gamma^{s+1}\cap Z_s)$. Therefore $\log_p(G : Z_{2s}) \leq eh^s + h$. Proposition 2 follows.

REFERENCES

[1] C. Casolo, U. Dardano and S. Rinauro, Variants of theorems of Baer and Hall on finite-by-hypercentral groups, J. Algebra **452**(2016), 279-287.

[2] G. A. Fernández-Alcober & M. Morigi, Generalizing a theorem of P. Hall on finite-by-nilpotent groups, Proc. Amer. Math. Soc. **137**(2009), 425-429.

[3] P. Hall, Finite-by-nilpotent groups, Proc. Cambridge Philos. Soc. **52**(1956), 611-616.

[4] O. H. Kegel and B. A. F. Wehrfritz, Locally Finite Groups, North-Holland Pub. Co., Amsterdam 1973.

[5] D. J. S. Robinson, Finiteness Conditions and Generalized Soluble Groups Vol. 1, Springer-Verlag, Berlin 1972.

[6] D. J. S. Robinson, A Course in the Theory of Groups, Springer-Verlag, New York 1982.

[7] B. A. F. Wehrfritz, Group and Ring Theoretic Properties of Polycyclic Groups, Springer-Verlag, London 2009.

[8] B. A. F. Wehrfritz, Variants of theorems of Schur, Baer and Hall, Ric. d. Mat., to appear.