Rend. Sem. Mat. Univ. Padova 1xx (201x) **Rendiconti del Seminario Matematico della Università di Padova**

A construction of a uniform continuous minimizing movement associated with a singular functional

Yoshihiko Yamaura

Abstract – A minimizing movement is constructed associated with a singular functional introduced by Alt and Caffarelli in order to study a free boundary problem. The main purpose of the present research is to construct a minimizing movement, which is uniformly continuous with respect to both time and space variables. The strategy is to regularize the singular term of time discretized functionals, and then to pass to the limit in the regularization parameter in the sense of Γ-convergence keeping the time discretization parameter fixed.

Mathematics Subject Classification (2010). 35J20

Keywords. Minimizing movement, free boundary problem, Γ-convergence.

1. Introduction

The goal of this paper is to construct a uniformly continuous minimizing movement for the singular functional introduced by Alt and Caffarelli [1] in order to investigate regularity properties of stationary free boundary. Before stating the main theorem of this paper, we present the definition of minimizing movement in a general metric space setting ([5]) together with introducing some terminology used throughout this paper.

Definition 1.1 (minimizing movement). Let X be a metric space with metric d, and $F : X \to [0, \infty]$ be a function. Let h be a positive number and suppose that $u_0 \in X$ is an element such that $F(u_0) < +\infty$. Set $u_{0,h} = u_0$, and recursively define $u_{n,h}$, $n \in \mathbb{N}$, as a minimizer of the variational problem:

$$\text{“ } \textit{Minimize } \frac{d(u, u_{n-1,h})^2}{h} + F(u) \textit{ among all } u \in X. \text{ ”}$$

Yoshihiko Yamaura, Department of Mathematics, Nihon University, 25-40 Sakurajosui 3-chome, Setagaya, Tokyo, Japan

E-mail: yamaura@math.chs.nihon-u.ac.jp

We call a sequence $(u_{n,h})_{n=0}^{\infty}$ defined in this way *the discrete minimizing movement* and designate by $\mathrm{DMM}(F; u_0, h)$ the collection of all such sequences. We say that $u = u(t) : [0,\infty) \to X$ is *a minimizing movement for F starting from u_0* if there exists a sequence $(h_j)_{j=1}^{\infty}$ converging to zero as $j \to \infty$ and $(u_{n,h_j})_{n=0}^{\infty} \in \mathrm{DMM}(F; u_0, h_j)$ such that

$$\lim_{j\to\infty} u_{h_j}(t) = u(t) \text{ in } X \tag{1.1}$$

for each $t \geq 0$, where $u_h(t) = u_h(t,x)$ for $(t,x) \in (t_{n-1}^{(h)}, t_n^{(h)}] \times \mathbb{R}^N$, $t_n^{(h)} = nh$, $n = 0, 1, 2, \ldots$. We call such a limit function $u(t)$ *the minimizing movement of F starting from u_0* and designate by $\mathrm{MM}(F; u_0)$ the collection of all such $u(t)$.

Although the term " *generalized* minimizing movement " is commonly used in references, the word " generalized " is omitted in this paper for brevity. Throughout this paper, " DMM " and " MM " will refer to " discrete minimizing movement " and " minimizing movement ", respectively.

Our main theorem is the following.

Theorem 1.2 (Main result). *Define a functional* $\mathrm{AC} : L^2(\mathbb{R}^N) \to [0,\infty]$ *as follows:*

$$\mathrm{AC}\,(u) := \begin{cases} \displaystyle\int_{\mathbb{R}^N} \Big(|\nabla u|^2 + \chi(u) \Big) dx & \textit{for } u \in W^{1,2}(\mathbb{R}^N) \\ +\infty & \textit{for } u \in L^2(\mathbb{R}^N) \setminus W^{1,2}(\mathbb{R}^N), \end{cases}$$

where $\chi(\cdot)$ is the characteristic function of the interval $(0,\infty)$. Suppose that the initial data u_0 satisfies conditions (2.6) of section 2. Then there exists an element $u = u(t) \in \mathrm{MM}\,(\mathrm{AC}\,; u_0)$ such that

(i) $0 \leq u \leq \sup_{\mathbb{R}^N} u_0$ *in* $[0,\infty) \times \mathbb{R}^N$,

(ii) *There exist positive constants $C_1 = C_1(u_0)$ and $C_2 = C_2(u_0, N)$ such that*

$$|u(t,x) - u(t,x')| \leq C_1 |x - x'| \quad \textit{for } t \geq 0,\ x, x' \in \mathbb{R}^N$$

and

$$|u(t,x) - u(t',x)| \leq C_2 |t - t'|^{\frac{1}{2}} \quad \textit{for } t, t' \geq 0,\ x \in \mathbb{R}^N.$$

A general existence result for MM is known.

Theorem 1.3 (Existence [3, Theorem 2.4]). *Let X be a metric space with metric d, and let $F : X \to [0,\infty]$ be a functional fulfilling the following conditions:*

(F1) *F is lower semicontinuous on X,*

(F2) *The set $\{u \in X \mid F(u) \leq k\}$ is compact in X for any positive constant k.*

Then, for any $u_0 \in X$ with $F(u_0) < +\infty$, there exists an element of $MM(F; u_0)$.

Set $F = \mathrm{AC}$ and $(X, d) = (L^2_{\mathrm{loc}}(\mathbb{R}^N), d_{L^2_{\mathrm{loc}}(\mathbb{R}^N)})$, and extend AC to be equal to $+\infty$ on $L^2_{\mathrm{loc}}(\mathbb{R}^N) \setminus L^2(\mathbb{R}^N)$. Here, the distance $d_{L^2_{\mathrm{loc}}(\mathbb{R}^N)})$ is defined for $f, g \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$ as

$$d_{L^2_{\mathrm{loc}}(\mathbb{R}^N)}(f, g) := \sum_{\ell=1}^{\infty} \frac{1}{2^\ell} \frac{||f-g||_{L^2(B_\ell(0))}}{1 + ||f-g||_{L^2(B_\ell(0))}},$$

where $B_\ell(0) = \{x \in \mathbb{R}^N \,|\, |x| < \ell\}$. Then, since the conditions (F1) and (F2) are fulfilled, the existence of an element of $\mathrm{MM}\,(\mathrm{AC}\,; u_0)$ is immediate from the above theorem. However, if we proceed the argument further along these lines, we will face the situation of investigating the regularity of MM having at our disposal only the fact that $u_{n,h}$ is a minimizer of a time discretized functional with a singular term. To obtain such regularity seems difficult since we can rely neither on the regularity theory for the stationary problem [1, 2] nor on the standard theory for parabolic partial differential equations. For this reason, we propose an approach formulated by adopting the local estimation technique in the framework of the minimizing movement scheme. Namely, we first establish a DMM with an equi-continuity property, and then construct a MM maintaining the regularity. More precisely, we carry out the construction according to the following plan:

(step 1) Regularize χ and establish equi-continuity for piecewise linear function generated by each minimizers.

(step 2) Pass to the limit as $\varepsilon \to 0$, while $h > 0$ is kept fixed. Here $\varepsilon > 0$ is a regularization parameter for χ, and $h > 0$ is the width of time discretization.

(step 3) Pass to the limit as $h \to 0$ to obtain the desired MM.

In the present paper, we concentrate on (step 2) and (step 3), whereas (step 1) has already been delivered in the preceding paper [26]. Thus, on combination of the results of the present paper and [26], we complete the proof of Theorem 1.2 according to the strategy above. Let us give an outline of the argument in each of the above all steps.

(step 1) is based on the local estimation for solutions to Euler-Lagrange equations of regularized functionals. Let $u^\varepsilon_{0,h} = u_0$ and $u^\varepsilon_{n,h}$, $n \in \mathbb{N}$, be a minimizer of the time-discretized and regularized functional

$$\mathrm{AC}^{\,\varepsilon}_{\,n,h}(u) = \frac{1}{h}||u - u^\varepsilon_{n-1,h}||^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \Big(|\nabla u|^2 + \chi_\varepsilon(u)\Big)dx,$$

where χ_ε, $\varepsilon > 0$, represents a regularization of χ such that $\chi_\varepsilon \to \chi$ as $\varepsilon \to 0$. Then it is shown that $\hat{u}^\varepsilon_h$, the piecewise linear function generated by $(u^\varepsilon_{n,h})^\infty_{n=0}$, is equi-continuous in $(0, \infty) \times \mathbb{R}^N$. In the proof, the techniques in the paper [11] are essentially exploited. The precise proof of (step 1) is given in the paper [26], whose results are summarized in section 2.

In (step 2), for the purpose of constructing a uniformly continuous DMM, we pass to the limit as $\varepsilon \to 0$ for each fixed $n \in \mathbb{N}$ and $h > 0$, preserving the regularity property. With the help of the result of (step 1), it is possible to find an infinitesimal (ε_j) such that $u^{\varepsilon_j}_{n,h}$ converges to a function $u_{n,h}$ locally uniformly on $\mathbb{R}^N$. By the definition of DMM, the limit function $u_{n,h}$ is required to be a minimizer of $\mathrm{AC}_{n,h}$ which is defined by replacing $u^{\varepsilon}_{n-1,h}$ and χ_ε in $\mathrm{AC}^{\varepsilon}_{n,h}$ by $u_{n-1,h}$ and χ, respectively. To this end, we only have to show that the sequence of functionals $\mathrm{AC}^{\varepsilon_j}_{n,h}$ converges, as $j \to \infty$, to $\mathrm{AC}_{n,h}$ in the sense of *Γ-convergence*, the variational convergence, in terms of the topology of $L^2(\mathbb{R}^N)$. This fact follows from the global $L^2(\mathbb{R}^N)$-convergence of $u^{\varepsilon_j}_{n-1,h}$ to $u_{n-1,h}$, which is shown by employing the comparison argument investigated in section 3. The Γ-convergence theory is originally introduced as an approximation method in the calculus of variation ([12], [8], [24]). We mention [6] and [7] as examples of its application to singular functionals. This theory is also applied to time evolutionary problems, as shown in this paper (cf. [14], [25]).

Finally, in (step 3), we derive the desired regular MM by passing to the limit as $h \to 0$, preserving again the regularity property. Having shown that $(\hat{u}_h)$ is equi-continuous in $\mathbb{R}^N \times (0, \infty)$, we can find an infinitesimal (h_j) such that $u_{h_j}(t)$ converges to $u(t)$ in the topology of $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ for each fixed $t > 0$. However, by the definition of MM, such a convergence is too weak to attain our final goal. Indeed, we are required to verify the convergence in terms of the global $L^2(\mathbb{R}^N)$-norm. This is accomplished again by invoking the comparison argument for the functional deprived of the χ-term. In fact, since the functional F without the χ-term is convex, we can take advantage of the monotonicity property of its subgradient, i.e., the Laplace operator ([3]).

Since the functional AC is singular because of χ-term, we can not adopt the local estimation technique directly. For this reason we regularize the singular term ([26]), so that we can exploit the technique in order to establish the regularity of each solutions. Once we pass to the limit in the regularization parameter, we can proceed our investigation according to the MM scheme. In this way, our analysis in this paper makes possible to apply the local estimation technique in the MM method through the regularization of functionals.

We illustrate the relation to other results and the background of our research. The time evolution corresponding to the stationary free boundary problem was treated by Caffarelli and Vázquez [11] for the problem of flame propagation. They find a function u and a domain D in $(0, \infty) \times \mathbb{R}^n$ such that $\partial_t u = \triangle u$ in D and $|\nabla_x u| = 1$ on ∂D. Their method is based on a singular perturbation of the heat equation by the derivative of a regularized characteristic function, and the investigation is performed essentially by choosing an initial data such that the solution has a monotonicity property.

Minimizing movement method was originally introduced by De Giorgi [3] in order to construct curves of maximal slope (see [20]) associated with singular energy functionals whose Euler-Lagrange equation can not be formulated. This approach is frequently adopted in recent works on PDEs, especially in the studies of time evolutionary problems with background in geometric measure theory (see [5] and the references therein). In fact, a minimizing movement turns out to be a curve of maximal slope if continuity is assumed for the energy functional and for its " slope ", which is a generalized notion of the modulus of gradient. Although a curve of maximal slope is a notion defined in a scalar setting because of the lack of differentiability of functionals, once the norm of a subdifferential to the functional equals to the slope, a " gradient flow equation " with a vectorial structure can be established ([5]). For instance, an application to an image segmentation problem defined on the space of functions of bounded variation has been reported ([4], [10]). In the reference [23], a curve of maximal slope to the functional AC is constructed in the one-dimensional case based on the method established in [20], which is a different formulation from the time discretization approach presented here.

The local estimation technique is known in regularity theory for weak solutions to elliptic and parabolic PDEs, like as, for instance, Campanato theory ([9, 13]). It has been developed by Kikuchi [19] to apply such a local estimation technique to MM in terms of energy functionals whose Euler-Lagrange equations can be formulated. The goal of this approach is to investigate equi-regularity of approximate solutions generated by time-discretized PDEs of elliptic type. It is worth mentioning that the aim of studying equi-regularity of approximate solutions is different from the purposes of usual regularity theory for the weak solutions to parabolic PDEs. The investigation of regularity for approximate solutions with an approximation parameter is inevitably divided into several cases depending on the relation between the size of the considered local parabolic cylinder and the width of time discretization. Thus, extremely precise and careful calculations are indispensable to this analysis, which is one of the distinct features of this method (refer to [17, 18, 26]). By adopting this method together with the Campanato theory, a weak solution to a parabolic system with a locally uniform continuity under a considerably weak initial and boundary conditions is constructed in [16]. Since this method is also *constructive* in the sense that an approximate solution is defined by minimizers of variational functionals and satisfies the Euler-Lagrange equation, the researches in the numerical analysis for free boundary problems have been investigated through a regularization technique as treated in this paper (e.g. [15, 22]).

We explain about the relation between the MM constructed in this paper and the evolution problem. Although the MM can not be a solution of just a single parabolic equation in time-space domain because of the characteristic function term, it is expected to be shown to be a solution of time evolutional problem in the sense of a Curve of Maximal Slope (CMS henceforth) to the Alt-Caffarelli functional. But this problem is still open. As stated above, in [23] a CMS is constructed in the case of one dimension by a minimizing sheme different from ours, and it is proved to be a weak solution in the sense of [11]. In the proof, an appropriate restriction of the domain of functional makes possible to apply the general theory of [20] for constructing CMS. On the other hand, in Part I of [5], a sufficient condition that a MM become to be a CMS is proposed in general dimension. Since the condition is equivalent to that of [20], the MM of this paper is expected to be a CMS also in general dimension. However, to demonstrate this conjecture along the line of [23] is considered to be difficult. Indeed, in [23] the function at which the slope of the functional is finite can be explicitly described owing to the particularity of one dimension.

We end this introduction by stating the future prospects of the problem treated in this paper. Due to the existence of the characteristic function term, the uniqueness of the solution does not hold in this problem. Therefore, the MM constructed by the method of this paper is expected to have better properties compared with the weak solution of the paper [11], since the MM is defined through an energy minimizing process. The key of the regularity theory of free boundary in [1] is a Lipschitz regularity of minimizers. It is considered to be significant that we establish in this paper the corresponding regularity property in the time-evolutional setting by a minimizing scheme, although other tools as *non-degeneracy property* have not been proved yet. We also expect that by using the MM obtained in this paper the investigation the regularity of the time evolutional free boundary becomes possible without any monotonicity hypothesis as assumed in [11].

We list now some of the main notations which will be used throughout this paper.

Notation The letters $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the real integer and the natural number system, respectively. Let $(f_n)_{n=0}^{\infty}$ be a sequence of functions defined on $\mathbb{R}^N$, h a positive number, and $t_n^{(h)} = nh$ $(n \in \mathbb{Z})$. We set the function f_h defined in $\mathbb{R}^N \times (-h, \infty)$ as follows:

$$f_h(t,x) := f_n(x) \quad \text{for } (t,x) \in \mathbb{R}^N \times (t_{n-1}^{(h)}, t_n^{(h)}],\ n = 0, 1, \dots. \tag{1.2}$$

We call f_h *the h-step function generated by* (f_n). We also set the function $\hat{f}_h$ defined on $\mathbb{R}^N \times (0, \infty) \to \mathbb{R}$ as follows:

$$\begin{aligned} \hat{f}_h(t,x) := \frac{t_n^{(h)} - t}{h} f_{n-1}(x) + \frac{t - t_{n-1}^{(h)}}{h} f_n(x)& \\ \text{for } (t,x) \in [t_{n-1}^{(h)}, t_n^{(h)}] \times \mathbb{R}^N,\ n = 1, 2, \dots.& \end{aligned} \tag{1.3}$$

We call $\hat{f}_h$ *the h-piecewise linear function generated by* (f_n).

The N dimensional Lebesgue measure is denoted by $\mathcal{L}^N$. We use the standard notation for the Lebesgue and Sobolev spaces $L^p(\mathbb{R}^N)$ and $W^{m,p}(\mathbb{R}^N)$, $1 \le p \le \infty$, $m \in \mathbb{N}$, equipped with the norms $||\cdot||_{L^p(\mathbb{R}^N)}$, $||\cdot||_{W^{m,p}(\mathbb{R}^N)}$, respectively. $L^2_{\text{loc}}(\mathbb{R}^N)$ is the set of measurable functions whose square are locally integrable in $\mathbb{R}^N$. $C^1(\mathbb{R}^N)$ indicates the set of functions of C^1-class in $\mathbb{R}^N$, and $C^{1,\alpha}(\mathbb{R}^N)$, $0 < \alpha < 1$, is the space of $C^1(\mathbb{R}^N)$-functions whose derivatives are Hölder continuous with the exponent α. For a function f on $\mathbb{R}^N$, we designate by $\operatorname{spt} f$ the support of f, $\mathbb{R}^N \setminus \bigcup U$, where union is taken over all open sets in which $f = 0$ holds almost everywhere. For a positive number δ and a point a of $\mathbb{R}^N$, we use the notation $B_\delta(a)$ for the open ball in $\mathbb{R}^N$ of radius δ with center a. By (t, x) we denote the pair of variables t and x, where t is a time in $(-\infty, \infty)$ and $x = (x_1, x_2, \dots, x_N)$ a point of $\mathbb{R}^N$. We say that a sequence (ε_j) is *an infinitesimal sequence* if $\varepsilon_j > 0$ for

$j \in \mathbb{N}$ and $\varepsilon_j \to 0$ as $j \to \infty$. For a sequence (f_j) of functions defined on $A \subset \mathbb{R}^m$, $m \in \mathbb{N}$, and a function f on A, we write $f_j \rightrightarrows f$ on A to mean that f_j converges to f uniformly on A as $j \to \infty$. For a function f on $\mathbb{R}^N$, a subset A of $\mathbb{R}^N$ and a real number δ, we set $A(f > \delta) = \{x \in A \,|\, f(x) > \delta\}$. Other symbols $A(f \geq \delta)$, $A(f < \delta)$, $A(f \leq \delta)$, etc. are understood analogously.

2. Preliminaries

Let χ be the characteristic function of the interval $(0, \infty)$ of $\mathbb{R}$: $\chi(r) := 1$ for $r > 0$, and $:= 0$ for $r \leq 0$. For any $\varepsilon > 0$, we indicate by χ_ε a smooth approximation of χ which satisfies the following properties:

$$\chi_\varepsilon \in C^\infty(\mathbb{R}). \tag{2.1}$$

$$0 \leq \chi_\varepsilon \leq 1 \text{ in } \mathbb{R}. \tag{2.2}$$

$$\chi_\varepsilon' \geq 0 \text{ in } \mathbb{R}, \ \ \chi_\varepsilon' > 0 \text{ in } (0, \varepsilon). \tag{2.3}$$

$$\chi_\varepsilon(r) = \begin{cases} 0 & \text{for } r \in (-\infty, 0], \\ 1 & \text{for } r \in [\varepsilon, \infty). \end{cases} \tag{2.4}$$

$$|\chi_\varepsilon'| \leq \frac{K}{\varepsilon}, \ |\chi_\varepsilon''| \leq \frac{K}{\varepsilon^2} \ \text{ in } \mathbb{R} \text{ for some positive constant } K. \tag{2.5}$$

Suppose u_0 is a given initial datum fulfilling

$$\begin{cases} u_0 \in W^{1,2}(\mathbb{R}^N) \cap W^{2,\infty}(\mathbb{R}^N) \cap C^{1,\alpha}(\mathbb{R}^N) \ \text{ forsome } \alpha \in (0,1), \\ \mathcal{L}^N(\operatorname{spt} u_0) < +\infty, \\ 0 \leq u_0 \leq \sup\limits_{\mathbb{R}^N} u_0 < +\infty \text{ in } \mathbb{R}^N. \end{cases} \tag{2.6}$$

Then, in particular, we notice that $\mathrm{AC}\,(u_0) < +\infty$.

For arbitrarily fixed positive numbers h and ε, set $u^\varepsilon_{0,h} = u_0$ and recursively define $u^\varepsilon_{n,h}$ $(n = 1, 2, \ldots)$ as a minimizer in $L^2(\mathbb{R}^N)$ of the functional

$$\begin{aligned} &\mathrm{AC}^\varepsilon_{n,h}(u) = \\ &= \begin{cases} \displaystyle\int_{\mathbb{R}^N} \Big(\frac{|u - u^\varepsilon_{n-1,h}|^2}{h} + |\nabla u|^2 + \chi_\varepsilon(u) \Big)\, dx & \text{for } u \in W^{1,2}(\mathbb{R}^N) \\ +\infty & \text{for } u \in L^2(\mathbb{R}^N) \setminus W^{1,2}(\mathbb{R}^N). \end{cases} \end{aligned} \tag{2.7}$$

Remark 2.1 (Collection of sequences $\mathrm{ADMM}\,(\mathrm{AC}\,; u_0, \varepsilon, h)$). The existence of a minimizer of the variational problem above is demonstrated by the usual direct method in the calculus of variation. The energy comparison with the constant function of the value zero confirms that $\mathrm{AC}^\varepsilon_{n,h}(u^\varepsilon_{n,h}) < +\infty$ for $n = 1, 2, \ldots$, and, in particular, $u^\varepsilon_{n,h} \in W^{1,2}(\mathbb{R}^N)$. Since $\mathrm{AC}^\varepsilon_{n,h}$ is not convex, the minimizer is not necessarily unique. Accordingly, more than one sequence $(u^\varepsilon_{n,h})^\infty_{n=0}$ may be defined. For this reason, we set up the notation $\mathrm{ADMM}\,(\mathrm{AC}\,; u_0, \varepsilon, h)$ to mean the collection of all such sequences, where “ ADMM ” is the abbreviation of the expression Approximate Discrete Minimizing Movement.

Henceforth, for an arbitrarily fixed positive numbers h and ε, let $(u^{\varepsilon}_{n,h})^{\infty}_{n=0} \in$ ADMM$\,(\mathrm{AC}\,; u_0, \varepsilon, h)$ be given. We list up properties of $(u^{\varepsilon}_{n,h})$ established in [26] which will play an important role in this paper.

By a truncation argument, the following result follows from the third condition of (2.6).

Theorem 2.2 (Weak maximum principle : Theorem 2, [26]). *It holds that*

$$0 \le u^{\varepsilon}_{n,h} \le \sup_{\mathbb{R}^N} u_0 \qquad \text{in } \mathbb{R}^N \tag{2.8}$$

for $n = 0, 1, 2, \cdots$.

Theorem 2.3 (Uniform gradient bound Theorem 4, [26]). *It holds that*

$$||\nabla u^{\varepsilon}_{n,h}||_{L^\infty(\mathbb{R}^N)} \le C \max\left\{M^{\frac{1}{2}},\ ||\nabla u_0||_{L^\infty(\mathbb{R}^N)}\right\} =: L \tag{2.9}$$

for $n = 0, 1, 2, \ldots$, *where* $M = \sup_{\mathbb{R}}(\chi_1' + |\chi_1''|)$, *and* C *is a positive constant independent of* ε, h *and* n.

Here, we notice that the first condition of (2.6) on the initial data u_0 is needed to demonstrate Theorem 2.3 (see [26] for further details).

Corollary 2.4. *Let* u^{ε}_h *be the* h*-piecewise constant function generated by* $(u^{\varepsilon}_{n,h})^{\infty}_{n=0}$. *Then it holds that*

$$\sup_{(-h,\infty)\times\mathbb{R}^N} |\nabla u^{\varepsilon}_h| \le L,$$

where L *is the positive constant defined in* (2.9).

Theorem 2.5 (Hölder estimate Theorem 1, [26]). *Let* $\hat{u}^{\varepsilon}_h$ *be the piecewise linear function generated by* $(u^{\varepsilon}_{n,h})^{\infty}_{n=0}$. *Then there exists a positive constant* H *independent of* ε *and* h *such that*

$$|\hat{u}^{\varepsilon}_h(t, x_0) - \hat{u}^{\varepsilon}_h(t', x_0)| \le H|t - t'|^{\frac{1}{2}}$$

for $0 \le t < t' < +\infty$ *and* $x_0 \in \mathbb{R}^N$.

Additionally to the results listed above, we also use the following property:

Lemma 2.6. *It holds that*

$$\sup_{\substack{n=0,1,\ldots\\ \varepsilon,h>0}} ||\nabla u^{\varepsilon}_{n,h}||_{L^2(\mathbb{R}^N)} < +\infty.$$

PROOF. In view of the minimality of $u^\varepsilon_{n,h}$, we infer

$$\begin{aligned}\int_{\mathbb{R}^N}\Big(|\nabla u^\varepsilon_{n,h}|^2+\chi_\varepsilon(u^\varepsilon_{n,h})+\frac{|u^\varepsilon_{n,h}-u^\varepsilon_{n-1,h}|^2}{h}\Big)\,dx\le\\ \le\int_{\mathbb{R}^N}\Big(|\nabla u^\varepsilon_{n-1,h}|^2+\chi_\varepsilon(u^\varepsilon_{n-1,h})\Big)\,dx.\end{aligned}\tag{2.10}$$

Dropping the fractional term of the left-hand side, we, in particular, have

$$\int_{\mathbb{R}^N}\Big(|\nabla u^\varepsilon_{n,h}|^2+\chi_\varepsilon(u^\varepsilon_{n,h})\Big)\,dx\le\int_{\mathbb{R}^N}\Big(|\nabla u^\varepsilon_{n-1,h}|^2+\chi_\varepsilon(u^\varepsilon_{n-1,h})\Big)\,dx.$$

Utilizing this inequality iteratively, we deduce

$$\int_{\mathbb{R}^N}|\nabla u^\varepsilon_{n,h}|^2\,dx\le\int_{\mathbb{R}^N}\Big(|\nabla u_0|^2+\chi_\varepsilon(u_0)\Big)\,dx\le\int_{\mathbb{R}^N}|\nabla u_0|^2\,dx+\mathcal{L}^N(\operatorname{spt}u_0).$$

Since the last quantity is finite and independent of n,ε and h, we arrive at our conclusion. □

3. The comparison principle

In this section, we establish a comparison principle which will be an essential tool to show the convergence stated in the next section.

For given functions $v_0,w_0\in L^2(\mathbb{R}^N)$, define functionals F and G on $L^2(\mathbb{R}^N)$ as follows:

$$F(\phi):=\begin{cases}\displaystyle\int_{\mathbb{R}^N}\Big(\frac{|\phi-v_0|^2}{h}\,dx+|\nabla\phi|^2+\chi_\varepsilon(\phi)\Big)\,dx & \text{for } \phi\in W^{1,2}(\mathbb{R}^N)\\ +\infty & \text{for } \phi\in L^2(\mathbb{R}^N)\setminus W^{1,2}(\mathbb{R}^N),\end{cases}$$

and

$$G(\phi):=\begin{cases}\displaystyle\int_{\mathbb{R}^N}\Big(\frac{|\phi-w_0|^2}{h}\,dx+|\nabla\phi|^2\Big)\,dx & \text{for } \phi\in W^{1,2}(\mathbb{R}^N)\\ +\infty & \text{for } \phi\in L^2(\mathbb{R}^N)\setminus W^{1,2}(\mathbb{R}^N).\end{cases}$$

LEMMA 3.1 (Comparison). *Let v and w be minimizers of F and G, respectively. If $0\le v_0\le w_0$ in $\mathbb{R}^N$, then $0\le v\le w$ in $\mathbb{R}^N$.*

PROOF. The energy comparison with the constant function of the value zero tells us that v and w belong to $W^{1,2}(\mathbb{R}^N)$. In addition, the facts $v\ge0$ and $w\ge0$ in $\mathbb{R}^N$ are derived from the assumption $v_0,w_0\ge0$ in $\mathbb{R}^N$ by employing the usual truncation argument.

Let $A:=\{x\in\mathbb{R}^N\mid v(x)>w(x)\}$. Then our task is to verify $\mathcal{L}^N(A)=0$. We will proceed by contradiction. Assume that $\mathcal{L}^N(A)>0$. Putting $\overline{v}:=\min\{v,w\}$, we see $F(v) \leq F(\overline{v})$ by the minimality of v. Since $\overline{v} = v$ in $\mathbb{R}^N \setminus A$ and $\overline{v} = w$ in A, we deduce

$$F_A(v) \leq F_A(\overline{v}) = F_A(w), \tag{3.1}$$

where

$$F_A(\phi) = \int_A \Big(\frac{|\phi - v_0|^2}{h}\, dx + |\nabla \phi|^2 + \chi_\varepsilon(\phi) \Big)\, dx \qquad \text{for } \phi \in W^{1,2}(\mathbb{R}^N).$$

Analogously, we can verify

$$G_A(w) \leq G_A(v) \tag{3.2}$$

by using, in this case, the comparison function $\overline{w} := \max\{v, w\}$, where

$$G_A(\phi) = \int_A \Big(\frac{|\phi - v_0|^2}{h}\, dx + |\nabla \phi|^2 \Big)\, dx \qquad \text{for } \phi \in W^{1,2}(\mathbb{R}^N).$$

On account of (2.3), the inequality $\chi_\varepsilon(w) \leq \chi_\varepsilon(v)$ holds in A. In fact, the equality is shown to hold in the following manner: Due to the relations (3.1) and (3.2), we infer

$$\begin{aligned} &\int_A \Big(\frac{|v - v_0|^2}{h} + |\nabla v|^2 + \chi_\varepsilon(v) \Big)\, dx \leq \int_A \Big(\frac{|w - v_0|^2}{h} + |\nabla w|^2 + \chi_\varepsilon(w) \Big)\, dx = \\ &= \int_A \Big\{ \frac{|w - w_0|^2}{h} + |\nabla w|^2 + \chi_\varepsilon(w) + \frac{|w - v_0|^2}{h} - \frac{|w - w_0|^2}{h} \Big\}\, dx \leq \\ &\leq \int_A \Big\{ \frac{|v - w_0|^2}{h} + |\nabla v|^2 + \chi_\varepsilon(w) + \frac{|w - v_0|^2}{h} - \frac{|w - w_0|^2}{h} \Big\}\, dx. \end{aligned} \tag{3.3}$$

We thereby find

$$\begin{aligned} &\int_A \big\{ \chi_\varepsilon(v) - \chi_\varepsilon(w) \big\}\, dx \leq \\ &\leq \frac{1}{h} \int_A \Big\{ (|v - w_0|^2 - |v - v_0|^2) + (|w - v_0|^2 - |w - w_0|^2) \Big\}\, dx. \end{aligned} \tag{3.4}$$

The integrand on the right hand side of (3.4) equals to $2(v_0 - w_0)(v - w)$, which is nonpositive in A because $v > w$ and, by assumption, $v_0 \leq w_0$ in A. Thus from (3.4)

$$\int_A \big\{ \chi_\varepsilon(v) - \chi_\varepsilon(w) \big\}\, dx \leq 0.$$

As indicated above, however, $\chi_\varepsilon(v) - \chi_\varepsilon(w) \geq 0$ in A, and therefore it must hold that

$$\chi_\varepsilon(v) = \chi_\varepsilon(w) \tag{3.5}$$

in A. By (3.2) and (3.5) we have

$$\int_A \Big(\frac{|w-w_0|^2}{h} + |\nabla w|^2 + \chi_\varepsilon(w) \Big)\, dx \le \int_A \Big(\frac{|v-w_0|^2}{h} + |\nabla v|^2 + \chi_\varepsilon(v) \Big)\, dx,$$

and hence

$$F_A(w) \le F_A(v) + \frac{1}{h} \int_A \Big\{ (|v-w_0|^2 - |v-v_0|^2) + (|w-v_0|^2 - |w-w_0|^2) \Big\}\, dx.$$

The last integral coincides with the right-hand side of (3.4) which has already been proved to be nonpositive in A, and hence $F_A(w) \le F_A(v)$. Combining this with its reverse inequality (3.1), we eventually achieve

$$F_A(w) = F_A(v). \tag{3.6}$$

We are now prepared to derive a contradiction from our assumption $\mathcal{L}^N(A) > 0$. By setting $\tilde{v} := \frac{1}{2}(v + \overline{v})$, we shall verify the strict inequality $F(\tilde{v}) \lneqq F(v)$ which contradicts the minimality of v. For this, it is sufficient to show $F_A(\tilde{v}) \lneqq F_A(v)$ for $\tilde{v} = v$ in $\mathbb{R}^N \setminus A$, which is equivalent to the inequality

$$F_A(\tilde{v}) \lneqq \frac{F_A(v) + F_A(\overline{v})}{2} \tag{3.7}$$

by (3.6) and the fact $\overline{v} = w$ in A. Noticing the strict convexity of $|\cdot|^2$, we find

$$\begin{aligned} &\int_A \Big(\frac{|\tilde{v}-v_0|^2}{h} + |\nabla \tilde{v}|^2 \Big)\, dx \\ &\lneqq \frac{1}{2} \int_A \Big(\frac{|v-v_0|^2}{h} + |\nabla v|^2 \Big)\, dx + \frac{1}{2} \int_A \Big(\frac{|\overline{v}-v_0|^2}{h} + |\nabla \overline{v}|^2 \Big)\, dx \end{aligned} \tag{3.8}$$

by the fact $v \neq w$ in A and the assumption $\mathcal{L}^N(A) > 0$. If we have the convexity inequality

$$\int_A \chi_\varepsilon(\tilde{v})\, dx \le \frac{1}{2} \int_A \chi_\varepsilon(v)\, dx + \frac{1}{2} \int_A \chi_\varepsilon(\overline{v})\, dx, \tag{3.9}$$

we can attain the desired result (3.7) by adding (3.8) and (3.9). The inequality (3.9) is not generally expected to hold because of the lack of convexity of χ_ε. Nevertheless, in our particular situation, we can demonstrate the validity of the equality in the following manner. Recall that for almost every $x \in A$, it hold that $w(x) < v(x)$ and $\chi_\varepsilon(w(x)) = \chi_\varepsilon(v(x))$ by (3.5). Therefore, by taking into account the strict increasing property (2.3) of χ_ε on the interval $[0, \varepsilon]$, two cases are possible: either (a) $w(x) < v(x) \le 0$ or (b) $\varepsilon \le w(x) < v(x)$. However, since v and w are nonnegative in $\mathbb{R}^N$ as stated at the beginning of the proof, the possibility (a) is eliminated. We thus have $\varepsilon \le w = \overline{v} < \tilde{v} < v$ in A, and hence by (2.4) the identity $\chi_\varepsilon(\tilde{v}) = \chi_\varepsilon(v) = \chi_\varepsilon(\overline{v}) = 1$ in A, which, in particular, reveals that the values of both sides of (3.9) coincide. $\square$

REMARK 3.2. The assertion of Lemma 3.1 holds true even if we replace χ_ε by χ in the definition of F. Indeed, by such a replacement, the arguments related to χ_ε in the proof above remain valid.

4. The Γ-convergence of approximate functionals

Our eventual goal in this section is to show that $\mathrm{AC}^\varepsilon_{n,h}$ converges to $\mathrm{AC}_{n,h}$ as $\varepsilon \to 0$ in the sense of Γ-convergence. In order to establish this convergence, we need the convergence $u^\varepsilon_{n,h} \to u_{n,h}$ in the topology induced by the global norm $||\cdot||_{L^2(\mathbb{R}^N)}$.

LEMMA 4.1 (Passage to the limit $\varepsilon \to 0$). *Fix an arbitrary positive number h. For each $\varepsilon > 0$, choose an element $(u^\varepsilon_{n,h})_{n=0}^\infty \in \mathrm{ADMM}\,(\mathrm{AC}\,; u_0, \varepsilon, h)$. Then there exist an infinitesimal sequence (ε_j) and a sequence $(u_{n,h})_{n=0}^\infty \subset W^{1,2}(\mathbb{R}^N)$ such that for any fixed $n = 0, 1, 2, \dots$*

$$\text{(i)}\quad u^{\varepsilon_j}_{n,h} \rightrightarrows u_{n,h} \quad \textit{locally in } \mathbb{R}^N \quad (j \to \infty), \tag{4.1}$$

$$\text{(ii)}\quad u^{\varepsilon_j}_{n,h} \to u_{n,h} \quad \textit{in } L^2(\mathbb{R}^N) \quad (j \to \infty). \tag{4.2}$$

PROOF. In the case $n = 0$, by defining $u_{0,h} = u_0$ we have $u^\varepsilon_{0,h} = u_0 = u_{0,h}$ for any $\varepsilon > 0$, which especially produces the convergence (i) and (ii).

For $n \geq 1$, we know

$$\sup_{0<\varepsilon<1} \Big(||u^\varepsilon_{n,h}||_{L^\infty(\mathbb{R}^N)} + ||\nabla u^\varepsilon_{n,h}||_{L^\infty(\mathbb{R}^N)} \Big) < +\infty$$

by Theorem 2.2 and Theorem 2.3, and therefore it follows from the Ascoli-Arzelà theorem together with a diagonal argument that there exists an infinitesimal sequence (ε_j) and a sequence $(u_{n,h})_{n=1}^\infty \subset W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N)$ such that (i) holds.

We next show that the convergence (ii) holds if we take (ε_j) as above. Looking at (i), we readily find

$$u^{\varepsilon_j}_{n,h} \to u_{n,h} \quad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^N) \quad (j \to \infty) \tag{4.3}$$

for $n = 1, 2, \dots$ Let us prove the global convergence in $L^2(\mathbb{R}^N)$. For this purpose, we need an argument based on comparison. Put $v_{0,h} = u_0$, and define recursively $v_{n,h}$ $(n = 1, 2, 3, \dots)$ as a minimizer of the functional

$$G_{n,h}(v) = \begin{cases} \displaystyle\int_{\mathbb{R}^N} \Big(\frac{|v - v_{n-1,h}|^2}{h} + |\nabla v|^2 \Big)\, dx & \text{for } v \in W^{1,2}(\mathbb{R}^N) \\ +\infty & \text{for } v \in L^2(\mathbb{R}^N) \setminus W^{1,2}(\mathbb{R}^N). \end{cases}$$

By recursively applying Lemma 3.1 with $v_0 := u^\varepsilon_{n-1,h}$, $w_0 := v_{n-1,h}$, $F := \mathrm{AC}^{\,\varepsilon}_{n,h}$ and $G := G_{n,h}$, we deduce for $n = 1, 2, 3, \dots$

$$0 \leq u^\varepsilon_{n,h} \leq v_{n,h} \tag{4.4}$$

in $\mathbb{R}^N$. Insert ε_j into ε and let j go to infinity. Then we infer

$$0 \le u_{n,h} \le v_{n,h} \tag{4.5}$$

in $\mathbb{R}^N$. For a fixed positive number R,

$$\begin{aligned}\int_{\mathbb{R}^N} |u^{\varepsilon_j}_{n,h} - u_{n,h}|^2\,dx &= \int_{B_R(0)} |u^{\varepsilon_j}_{n,h} - u_{n,h}|^2\,dx + \int_{\mathbb{R}^N\setminus B_R(0)} |u^{\varepsilon_j}_{n,h} - u_{n,h}|^2\,dx \le \\ &\le \int_{B_R(0)} |u^{\varepsilon_j}_{n,h} - u_{n,h}|^2\,dx + 4\int_{\mathbb{R}^N\setminus B_R(0)} |v_{n,h}|^2\,dx\end{aligned} \tag{4.6}$$

by (4.4) and (4.5). Pass th the limit as $j \to \infty$ in (4.6). Then, the first term of the right-hand side is annihilated by (4.3), and as a result the next relation holds:

$$\varlimsup_{j\to\infty} \int_{\mathbb{R}^N} |u^{\varepsilon_j}_{n,h} - u_{n,h}|^2\,dx \le 4\int_{\mathbb{R}^N\setminus B_R(0)} |v_{n,h}|^2\,dx. \tag{4.7}$$

By letting R go to infinity, the right-hand side of (4.7) converges to zero because $v_{n,h}$ belongs to the class $L^2(\mathbb{R}^N)$. Hence we find

$$\lim_{j\to\infty} \int_{\mathbb{R}^N} |u^{\varepsilon_j}_{n,h} - u_{n,h}|^2\,dx = 0.$$

To complete our proof, we shall observe that $u_{n,h}$ belongs to the regularity class $W^{1,2}(\mathbb{R}^N)$. The inequality (4.5) and the fact that $v_{n,h}$ belongs to $L^2(\mathbb{R}^N)$ informs us that $u_{n,h}$ belongs to $L^2(\mathbb{R}^N)$. Moreover, by Lemma 2.6, $\sup_{j\in\mathbb{N}} ||\nabla u^{\varepsilon_j}_{n,h}||_{L^2(\mathbb{R}^N)} < +\infty$. Accordingly, by taking a subsequence, if necessary, there exists a function $w_i \in L^2(\mathbb{R}^N)$ $(i = 1, 2, \ldots, N)$ such that $\nabla_i u^{\varepsilon_j}_{n,h} \to w_i$ weakly in $L^2(\mathbb{R}^N)$ as $j \to \infty$. We can show that $w_i = \nabla_i u_{n,h}$ in $\mathbb{R}^N$, and hence we come to the conclusion $\nabla u_{n,h} \in L^2(\mathbb{R}^N)$. □

We are now ready to prove the Γ-convergence $\mathrm{AC}^{\varepsilon}_{n,h} \to \mathrm{AC}_{n,h}$ which is the final goal of this section. Let (ε_j) be an infinitesimal sequence determined as in Lemma 4.1 and let $(u_{n,h})$ be the sequence of the limit functions. Then, we define the functionals $\mathrm{AC}_{\,n,h}$ as follows:

$$\mathrm{AC}_{\,n,h}(u) = \\ = \begin{cases} \displaystyle\int_{\mathbb{R}^N} \Big(\frac{|u - u_{n-1,h}|^2}{h} + |\nabla u|^2 + \chi(u) \Big)\,dx & \text{for } u \in W^{1,2}(\mathbb{R}^N)\ , \\ +\infty & \text{for } u \in L^2(\mathbb{R}^N) \setminus W^{1,2}(\mathbb{R}^N). \end{cases}$$

Based on the convergence result of $(u^{\varepsilon_j}_{n,h})$ shown in Lemma 4.1, we shall demonstrate the desired result.

Proposition 4.2 (Γ-convergence). *It holds that*

$$\mathrm{AC}^{\varepsilon_j}_{n,h} \quad \longrightarrow \quad \mathrm{AC}_{n,h} \quad \textit{in the sense of } \Gamma(L^2(\mathbb{R}^N)) \quad (j \to \infty).$$

Proof. By definition, it is sufficient to show the following lower- and upper-semicontinuity property:

(lsc): *For any sequence* $(v_j)_{j=1}^{\infty} \subset L^2(\mathbb{R}^N)$ *and* $v \in L^2(\mathbb{R}^N)$ *with* $v_j \to v$ *in* $L^2(\mathbb{R}^N)$, *it holds*

$$\mathrm{AC}_{n,h}(v) \leq \varliminf_{j\to\infty} \mathrm{AC}^{\varepsilon_j}_{n,h}(v_j). \tag{4.8}$$

(usc): *For any* $v \in L^2(\mathbb{R}^N)$, *there exists a sequence* $(v_j)_{j=1}^{\infty} \subset L^2(\mathbb{R}^N)$ *with* $v_j \to v$ *in* $L^2(\mathbb{R}^N)$ *such that*

$$\varlimsup_{j\to\infty} \mathrm{AC}^{\varepsilon_j}_{n,h}(v_j) \leq \mathrm{AC}_{n,h}(v). \tag{4.9}$$

We first show (lsc). Choose a sequence $(v_j) \subset L^2(\mathbb{R}^N)$ and a function $v \in L^2(\mathbb{R}^N)$ arbitrarily such that $v_j \to v$ in $L^2(\mathbb{R}^N)$. Since in case $\varliminf_{j\to\infty} \mathrm{AC}^{\varepsilon_j}_{n,h}(v_j) = +\infty$, the conclusion surely holds, we may assume $\varliminf_{j\to\infty} \mathrm{AC}^{\varepsilon_j}_{n,h}(v_j) < +\infty$. Then we can choose a subsequence $(v_{j_\ell}) \subset (v_j)$ such that

$$\lim_{\ell\to\infty} \mathrm{AC}^{\varepsilon_{j_\ell}}_{n,h}(v_{j_\ell}) = \varliminf_{j\to\infty} \mathrm{AC}^{\varepsilon_j}_{n,h}(v_j) < +\infty. \tag{4.10}$$

For sufficiently large number ℓ, the value $\mathrm{AC}^{\varepsilon_{j_\ell}}_{n,h}(v_{j_\ell})$ is finite. Hence, by the definition of $\mathrm{AC}^{\varepsilon}_{n,h}$, it turns out that v_{j_ℓ} belongs to $W^{1,2}(\mathbb{R}^N)$, and the following relation holds:

$$\mathrm{AC}^{\varepsilon_{j_\ell}}_{n,h}(v_{j_\ell}) = \int_{\mathbb{R}^N} \Big(\frac{|v_{j_\ell} - u^{\varepsilon_{j_\ell}}_{n-1,h}|^2}{h} + |\nabla v_{j_\ell}|^2 + \chi_{\varepsilon_{j_\ell}}(v_{j_\ell}) \Big)\, dx$$

for $\ell \in \mathbb{N}$. We infer from (4.10) $\sup_{\ell\in\mathbb{N}} \mathrm{AC}^{\varepsilon_{j_\ell}}_{n,h}(v_{j_\ell}) < +\infty$ to obtain

$$\sup_{\ell\in\mathbb{N}} \int_{\mathbb{R}^N} |\nabla v_{j_\ell}|^2\, dx < +\infty.$$

Thus, up to an extraction of a suitable subsequence, we can suppose that

$$\begin{cases} \text{(a) } v_{j_\ell} \to v \quad \text{in } L^2(\mathbb{R}^N) \\ \text{(b) } v_{j_\ell} \to v \quad \text{a.e. in } \mathbb{R}^N \\ \text{(c) } \nabla v_{j_\ell} \rightharpoonup \nabla v \quad \text{weakly in } L^2(\mathbb{R}^N) \end{cases} \tag{4.11}$$

as $\ell \to \infty$ for some $v \in W^{1,2}(\mathbb{R}^N)$. Since $v \in W^{1,2}(\mathbb{R}^N)$, we have the relation

$$\mathrm{AC}_{\,n,h}(v) = \int_{\mathbb{R}^N} \Big(\frac{|v - u_{n-1,h}|^2}{h} + |\nabla v|^2 + \chi(v) \Big)\, dx.$$

Therefore the desired inequality (4.8) is rewritten as

$$\begin{aligned} &\int_{\mathbb{R}^N} \Big(\frac{|v - u_{n-1,h}|^2}{h} + |\nabla v|^2 + \chi(v) \Big)\, dx \le \\ &\le \varliminf_{\ell\to\infty} \int_{\mathbb{R}^N} \Big(\frac{|v_{j_\ell} - u^{\varepsilon_{j_\ell}}_{n-1,h}|^2}{h} + |\nabla v_{j_\ell}|^2 + \chi_{j_{\varepsilon_\ell}}(v_{j_\ell}) \Big)\, dx, \end{aligned} \tag{4.12}$$

which is shown as follows. In light of the convergence (c),

$$\int_{\mathbb{R}^N} |\nabla v|^2\, dx \le \varliminf_{\ell\to\infty} \int_{\mathbb{R}^N} |\nabla v_{j_\ell}|^2\, dx. \tag{4.13}$$

By applying Lemma 4.4 below to $E = B_R(0)$, we verify

$$\int_{B_R(0)} \chi(v)\, dx \le \varliminf_{\ell\to\infty} \int_{B_R(0)} \chi_{\varepsilon_{j_\ell}}(v_{j_\ell})\, dx \le \varliminf_{\ell\to\infty} \int_{\mathbb{R}^N} \chi_{\varepsilon_{j_\ell}}(v_{j_\ell})\, dx.$$

By letting $R \uparrow \infty$ we get

$$\int_{\mathbb{R}^N} \chi(v)\, dx \le \varliminf_{\ell\to\infty} \int_{\mathbb{R}^N} \chi_{\varepsilon_{j_\ell}}(v_{j_\ell})\, dx. \tag{4.14}$$

However, by the L^2-strong convergence (a) of (4.13) and (4.2), we derive for any $R > 0$

$$\begin{aligned} \int_{B_R(0)} \frac{|v - u_{n-1,h}|^2}{h}\, dx &= \lim_{\ell\to\infty} \int_{B_R(0)} \frac{|v_{j_\ell} - u^{\varepsilon_{j_\ell}}_{n-1,h}|^2}{h}\, dx \le \\ &\le \varliminf_{\ell\to\infty} \int_{\mathbb{R}^N} \frac{|v_{j_\ell} - u^{\varepsilon_{j_\ell}}_{n-1,h}|^2}{h}\, dx. \end{aligned}$$

Letting $R \to \infty$, we have

$$\int_{\mathbb{R}^N} \frac{|v - u_{n-1,h}|^2}{h}\, dx \le \varliminf_{\ell\to\infty} \int_{\mathbb{R}^N} \frac{|v_{j_\ell} - u^{\varepsilon_{j_\ell}}_{n-1,h}|^2}{h}\, dx. \tag{4.15}$$

The inequalities (4.13), (4.14) and (4.15) imply that the inequality (4.12) holds true.

Next we shall prove that (usc) is valid by setting $v_j := v$ for any $j \in \mathbb{N}$. If $\mathrm{AC}_{\,n,h}(v) = +\infty$, then the assertion is surely satisfied, and therefore it is not restrictive to consider the case $\mathrm{AC}_{\,n,h}(v) < +\infty$. In this case, since $v \in W^{1,2}(\mathbb{R}^N)$, it follows that

$$\mathrm{AC}^{\,\varepsilon_j}_{\,n,h}(v_j) = \mathrm{AC}^{\,\varepsilon_j}_{\,n,h}(v) \le \int_{\mathbb{R}^N} \Big(\frac{|v - u^{\varepsilon_j}_{n-1,h}|^2}{h} + |\nabla v|^2 + \chi(v) \Big)\, dx$$

by making use of the inequality $\chi_{\varepsilon_j}(r) \leq \chi(r)$ for $r \in \mathbb{R}^N$. Thereby, by passing to the limits as $j \to \infty$, we establish (4.9) with the help of the global convergence (4.2) in $L^2(\mathbb{R}^N)$. $\square$

Remark 4.3. We remark that the proof of (lsc) continues to be valid even if we use the local L^2-convergence property $u^{\varepsilon_j}_{n,h} \to u_{n,h}$. On the other hand, to accomplish the proof of (usc) the global, not local, L^2-convergence property (4.2) is decisive.

Lemma 4.4. *Let $E \subset \mathbb{R}^N$ be a measurable set with $\mathcal{L}^N(E) < +\infty$, and let (ε_ℓ) be an infinitesimal sequence. Suppose that $w_\ell \to w$ in $L^2(E)$ as $\ell \to \infty$. Then it holds that*

$$\int_E \chi(w)\,dx \leq \varliminf_{\ell\to\infty} \int_E \chi_{\varepsilon_\ell}(w_\ell)\,dx.$$

Proof. It is sufficient to show

$$\lim_{\ell\to\infty} \int_{E(w>0)} |\chi_{\varepsilon_\ell}(w_\ell) - \chi(w)|\,dx = 0. \tag{4.16}$$

Indeed, (4.16) yields

$$\begin{aligned}\varliminf_{\ell\to\infty} \int_E \chi_{\varepsilon_\ell}(w_\ell)\,dx \geq \lim_{\ell\to\infty} \int_{E(w>0)} \chi_{\varepsilon_\ell}(w_\ell)\,dx =& \\ = \int_{E(w>0)} \chi(w)\,dx = \int_E \chi(w)\,dx,&\end{aligned}$$

which is our conclusion.

Let us now prove (4.16). Since by (2.4) it holds that $\chi_{\varepsilon_\ell}(w_\ell) - \chi(w) = 1 - 1 = 0$ on $E(w>0) \cap E(w_\ell \geq \varepsilon_\ell)$, we have

$$\begin{aligned}\int_{E(w>0)} |\chi_{\varepsilon_\ell}(w_\ell) - \chi(w)|\,dx &= \int_{E(w>0)\cap E(w_\ell<\varepsilon_\ell)} \big|\chi_{\varepsilon_\ell}(w_\ell) - \chi(w)\big|\,dx \\ &\leq \mathcal{L}^N\big(E(w>0)\cap E(w_\ell<\varepsilon_\ell)\big).\end{aligned} \tag{4.17}$$

Given an arbitrary positive number δ, by the assumption $\mathcal{L}^N(E) < +\infty$, there exists a positive number σ such that

$$\mathcal{L}^N\big(E(0<w<2\sigma)\big) < \frac{\delta}{2}. \tag{4.18}$$

Furthermore, since $w_\ell \to w$ in $L^2(E)$, we can find $\ell_0 \in \mathbb{N}$ such that for $\ell > \ell_0$

$$\mathcal{L}^N\big(E(|w_\ell - w| > \sigma)\big) < \frac{\delta}{2}. \tag{4.19}$$

Let us choose $\ell_1 \in \mathbb{N}$ such that the following two conditions hold: " $\ell_1 > \ell_0$ " and " $\ell > \ell_1$ implies $\varepsilon_\ell < \sigma$ ". By (4.18) and (4.19), if $\ell > \ell_1$,

$$\begin{aligned}
&\mathcal{L}^N\big(E(w>0)\cap E\big(w_\ell<\varepsilon_\ell\big)\big)\\
&\le \mathcal{L}^N\big(E(w>0)\cap E\big(w_\ell<\sigma\big)\big)\\
&= \mathcal{L}^N\big(E(0<w<2\sigma)\cap E\big(w_\ell<\sigma\big)\big)+\mathcal{L}^N\big(E(w\ge 2\sigma)\cap E\big(w_\ell<\sigma\big)\big)\\
&\le \mathcal{L}^N\big(E(0<w<2\sigma)\big)+\mathcal{L}^N\big(E(|w_\ell-w|>\sigma)\big)<\delta.
\end{aligned}$$

Combining this estimate with (4.17) we arrive at (4.16). □

5. The proof of Theorem 1.2

The proof is split in four steps.

[Step 1] Review of the regularity properties of $(\hat{u}_h^\varepsilon)$.

Let u_0 be given as in (2.6), h and ε positive numbers, $(u_{n,h}^\varepsilon) \in \mathrm{ADMM}\,(\mathrm{AC}\,; u_0, \varepsilon, h)$, and $\hat{u}_h^\varepsilon$ the h-piecewise linear function generated by $(u_{n,h}^\varepsilon)$. Then by Theorem 2.2, 2.3 and 2.5, it holds that

$$\text{(5.1)}\qquad \begin{cases} 0 \le \hat{u}_h^\varepsilon \le \sup\limits_{\mathbb{R}^N} u_0 & \text{in } \mathbb{R}^N \times [0,\infty),\\ \big|\hat{u}_h^\varepsilon(t_0,x)-\hat{u}_h^\varepsilon(t_0,x')\big| \le L|x-x'| & \text{for } t_0 \ge 0,\ x,x' \in \mathbb{R}^N,\\ \big|\hat{u}_h^\varepsilon(t,x_0)-\hat{u}_h^\varepsilon(t',x_0)\big| \le H|t-t'|^{\frac{1}{2}} & \text{for } t,t' \ge 0,\ x_0 \in \mathbb{R}^N. \end{cases}$$

[Step 2] The construction of $(u_{n,h}) \in \mathbf{DMM}\,(\mathbf{AC}\,; u_0, h)$

Fix $h > 0$ arbitrarily. Let (ε_j) be an infinitesimal sequence determined as in Lemma 4.1 and let $(u_{n,h})$ be the sequence of the limit functions. We denote by $\hat{u}_h$ the h-piecewise linear function generated by $(u_{n,h})$. Let us demonstrate the following two facts [Fact 1] and [Fact 2]:

[Fact 1] $\hat{u}_h$ are equi-bounded and equi-continuous.

Let $(t,x) \in [0,\infty) \times \mathbb{R}^N$ and let n be a positive integer such that $t \in [t_{n-1}^{(h)}, t_n^{(h)})$. Then by (4.1) $u_{n,h}^{\varepsilon_j}$ converges to $u_{n,h}$ as $j \to \infty$ everywhere in $\mathbb{R}^N$, and hence

$$\begin{aligned}
\hat{u}_h^{\varepsilon_j}(t,x) &= \frac{(t_n^{(h)}-t)u_{n-1,h}^{\varepsilon_j}(x)+(t-t_{n-1}^{(h)})u_{n,h}^{\varepsilon_j}(x)}{h}\\
&\to \frac{(t_n^{(h)}-t)u_{n-1,h}(x)+(t-t_{n-1}^{(h)})u_{n,h}(x)}{h} = \hat{u}_h(t,x) \qquad \text{as } j \to \infty.
\end{aligned}$$

Thereupon, $\hat{u}_h^{\varepsilon_j}$ turns out to converge to $\hat{u}_h$ as $j \to \infty$ everywhere in $[0,\infty) \times \mathbb{R}^N$. Exploiting this convergence, we can passage to the limit as $j \to \infty$ in (5.1) with

$\varepsilon = \varepsilon_j$ to find

$$
\begin{cases}
0 \le \hat{u}_h \le \sup\limits_{\mathbb{R}^N} u_0 \quad \text{in } [0,\infty)\times\mathbb{R}^N, \\
\big|\hat{u}_h(t_0,x) - \hat{u}_h(t_0,x')\big| \le L|x-x'| \quad \text{for } t_0 \ge 0,\ x,x' \in \mathbb{R}^N, \\
\big|\hat{u}_h(t,x_0) - \hat{u}_h(t',x_0)\big| \le H|t-t'|^{\frac{1}{2}} \quad \text{for } t,t' \ge 0,\ x_0 \in \mathbb{R}^N.
\end{cases}
\tag{5.2}
$$

[Fact 2] The limit function $u_{n,h}$ is a minimizer of $\mathrm{AC}_{n,h}$ in $L^2(\mathbb{R}^N)$.

$u^{\varepsilon_j}_{n,h}$ is a minimizer of $\mathrm{AC}^{\varepsilon_j}_{n,h}$ in $L^2(\mathbb{R}^N)$, and $u^{\varepsilon_j}_{n,h} \to u_{n,h}$ in $L^2(\mathbb{R}^N)$ as shown in section 4. Therefore, the Γ-convergence $\mathrm{AC}^{\varepsilon_j}_{n,h} \to \mathrm{AC}_{n,h}$ leads us to the fact that the limit function $u_{n,h}$ minimizes the limit functional $\mathrm{AC}_{n,h}$ in $L^2(\mathbb{R}^N)$.

[Step 3] The construction of an element of MM (AC ; u_0)

Since from (5.2) the functions $\hat{u}_h$ are equi-bounded and equi-continuous in $[0,\infty)\times\mathbb{R}^N$ due to the Ascoli-Arzelà theorem, there exists an infinitesimal sequence (h_j) and a function u continuous in $[0,\infty)\times\mathbb{R}^n$ such that

$$
\hat{u}_{h_j} \rightrightarrows u \quad \text{locally in } [0,\infty)\times\mathbb{R}^N. \tag{5.3}
$$

We claim that this limit function $u = u(t)$ belongs to $\mathrm{MM}\,(\mathrm{AC}\,; u_0)$. To this aim, recall the Definition 1.1 of MM. Having already shown that $u_{n,h}$ is a minimizer of the functional $\mathrm{AC}_{n,h}$ in $L^2(\mathbb{R}^N)$, we only have to prove

$$
\begin{aligned}
&\text{(a)}\ \ u(t_0) \in L^2(\mathbb{R}^N) \\
&\text{(b)}\ \ u_{h_j}(t_0) \to u(t_0) \ \text{ in } L^2(\mathbb{R}^N) \quad (j\to\infty)
\end{aligned}
\tag{5.4}
$$

for each $t_0 \ge 0$. Fixing $t_0 \ge 0$ arbitrarily, we shall show (5.4) according to the following two steps.

(step 1) First we show that the desired convergence holds in terms of the topology of $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ weaker than $L^2(\mathbb{R}^N)$:

$$
u_{h_j}(t_0) \to u(t_0) \quad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^N). \tag{5.5}
$$

As for $\hat{u}_{h_j}$, we deduce from (5.3) that

$$
\hat{u}_{h_j}(t_0) \rightrightarrows u(t_0) \quad \text{locally in } \mathbb{R}^N,
$$

and therefore we readily deduce

$$
\hat{u}_{h_j}(t_0) \to u(t_0) \quad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^N). \tag{5.6}
$$

Consequently, we only have to show that the convergence (5.6) remains valid by replacing $\hat{u}_{h_j}(t_0)$ with $u_{h_j}(t_0)$. To this end, we carry out an estimation of the value of $||\hat{u}_h(t_0) - u_h(t_0)||_{L^2(\mathbb{R}^N)}$. Choose $n \in \mathbb{N}$ such that $t_0 \in (t_{n-1}^{(h)}, t_n^{(h)}]$. Then we see

(5.7)
$$|\hat{u}_h(t_0,x) - u_h(t_0,x)| \le \frac{|t_n^{(h)} - t_0|}{h}|u_{n,h}(x) - u_{n-1,h}(x)| \le |u_{n,h}(x) - u_{n-1,h}(x)|,$$

where we use $h = t_n^{(h)} - t_{n-1}^{(h)}$. On the other hand, since $u_{n,h}$ is a minimizer of $\mathrm{AC}_{n,h}$, by choosing $u_{n-1,h} \in W^{1,2}(\mathbb{R}^N)$ as an energy comparison function, we observe that

(5.8)
$$\int_{\mathbb{R}^N} \Big(\frac{|u_{n,h} - u_{n-1,h}|^2}{h} + |\nabla u_{n,h}|^2 + \chi(u_{n,h})\Big)\, dx \le \int_{\mathbb{R}^N} \Big(|\nabla u_{n-1,h}|^2 + \chi(u_{n-1,h})\Big)\, dx,$$

which particularly turns to the inequality $\mathrm{AC}\,(u_{n,h}) \le \mathrm{AC}\,(u_{n-1,h})$. By applying this inductively,

$$\mathrm{AC}\,(u_{n,h}) \le \mathrm{AC}\,(u_{0,h}) = \mathrm{AC}\,(u_0),$$

where we notice that the last quantity is a finite constant, independent of n and h, by the hypothesis (2.6). Dropping the last two terms of the left-hand side of (5.8), we get

$$\int_{\mathbb{R}^N} |u_{n,h} - u_{n-1,h}|^2\, dx \le \mathrm{AC}\,(u_0)h. \tag{5.9}$$

In this way, we infer from (5.7) and (5.9) that

$$||\hat{u}_h(t_0) - u_h(t_0)||_{L^2(\mathbb{R}^N)} \le \mathrm{AC}\,(u_0)^{\frac{1}{2}} h^{\frac{1}{2}}. \tag{5.10}$$

Passing to the limit as $j \to \infty$ in (5.10) with $h = h_j$, we observe that

$$\lim_{j\to\infty} ||\hat{u}_{h_j}(t) - u_{h_j}(t)||_{L^2(\mathbb{R}^N)} = 0.$$

Combining this with (5.6), we conclude (5.5).

(step 2) We claim that the convergence (5.5) continues to hold in terms of the topology of $L^2(\mathbb{R}^N)$ stronger than that of $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. In order to show this, we resort to the following comparison argument. Define

$$F(u) := \begin{cases} \int_{\mathbb{R}^N} |\nabla u|^2\, dx & \text{for } u \in W^{1,2}(\mathbb{R}^N) \\ +\infty & \text{for } u \in L^2(\mathbb{R}^N) \setminus W^{1,2}(\mathbb{R}^N), \end{cases} \tag{5.11}$$

and set $v_{0,h} := u_0$ for $h > 0$. Further, recursively define $v_{n,h} \in L^2(\mathbb{R}^N)$ $(n \ge 1)$ as a minimizer of the functional

$$F_{n,h}(u) := \int_{\mathbb{R}^N} \frac{|u - u_{n-1,h}|^2}{h}\, dx + F(u) \quad (u \in L^2(\mathbb{R}^N)).$$

We can show that $v_{n,h}$ belongs to $W^{1,2}(\mathbb{R}^N)$ by the argument of energy comparison with $w = 0$. With the aid of Remark 3.2, we find $0 \le u_{n,h} \le v_{n,h}$ in $\mathbb{R}^N$ $(n = 0, 1, 2, \ldots)$. Hence, by the definition of piecewise constant function we see

$$0 \le u_h(t_0) \le v_h(t_0) \quad \text{in } \mathbb{R}^N. \tag{5.12}$$

Taking advantage of the convexity of F, it turns out that there exists a function $v(t_0) \in L^2(\mathbb{R}^N)$ such that without selecting a subsequence

$$v_h(t_0) \to v(t_0) \quad \text{in } L^2(\mathbb{R}^N) \tag{5.13}$$

as $h \downarrow 0$ (refer to [3, Page 205, Esempio 2.1]). Let R be an arbitrary positive number. Then by (5.12)

$$\begin{aligned}
&\int_{\mathbb{R}^N \setminus B_R(0)} |u_h(t_0) - u(t_0)|^2 \, dx \le \\
&\le 2 \int_{\mathbb{R}^N \setminus B_R(0)} |u_h(t_0)|^2 \, dx + 2 \int_{\mathbb{R}^N \setminus B_R(0)} |u(t_0)|^2 \, dx \le \\
&\le 2 \int_{\mathbb{R}^N \setminus B_R(0)} |v_h(t_0)|^2 \, dx + 2 \int_{\mathbb{R}^N \setminus B_R(0)} |u(t_0)|^2 \, dx \le \\
&\le 4 \int_{\mathbb{R}^N} |v_h(t_0) - v(t_0)|^2 \, dx + 4 \int_{\mathbb{R}^N \setminus B_R(0)} \Big(|u(t_0)|^2 + |v(t_0)|^2 \Big) \, dx.
\end{aligned}$$

Now, set here $h = h_j$ and pass to the limit as $j \to \infty$ in the last relation. Then by (5.13) we obtain

$$\varlimsup_{j \to \infty} \int_{\mathbb{R}^N \setminus B_R(0)} |u_{h_j}(t_0) - u(t_0)|^2 \, dx \le 4 \int_{\mathbb{R}^N \setminus B_R(0)} \Big(|u(t_0)|^2 + |v(t_0)|^2 \Big) \, dx. \tag{5.14}$$

In view of (5.5) and (5.13), there exists a subsequence (h_{j_k}) of (h_j) such that $u_{h_{j_k}}(t_0) \to u(t_0)$ and $v_{h_{j_k}}(t_0) \to v(t_0)$ almost everywhere in $\mathbb{R}^N$ as $k \to \infty$. Now, by passing to the limit as $k \to \infty$ in (5.12) with h replaced by h_{j_k}, we achieve the inequality $0 \le u(t_0) \le v(t_0)$ in $\mathbb{R}^N$ which reveals to the assertion (a) of (5.4), since $v(t_0) \in L^2(\mathbb{R}^N)$. Utilizing the last inequality again, we obtain from (5.14)

$$\varlimsup_{j \to \infty} \int_{\mathbb{R}^N \setminus B_R(0)} |u_{h_j}(t_0) - u(t_0)|^2 \, dx \le 8 \int_{\mathbb{R}^N \setminus B_R(0)} |v(t_0)|^2 \, dx.$$

Invoking again the fact that $v(t_0)$ belongs to $L^2(\mathbb{R}^N)$, the last inequality informs us that for any positive number δ, there exists a positive number R_δ such that

$$\varlimsup_{j \to \infty} \int_{\mathbb{R}^N \setminus B_{R_\delta}(0)} |u_{h_j}(t_0) - u(t_0)|^2 \, dx < \delta. \tag{5.15}$$

Passing to the limit as $j \to \infty$ in the relation

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u_{h_j}(t_0) - u(t_0)|^2 \, dx = \\
&= \int_{B_{R_\varepsilon}(0)} |u_{h_j}(t_0) - u(t_0)|^2 \, dx + \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |u_{h_j}(t_0) - u(t_0)|^2 \, dx,
\end{aligned}$$

we deduce from (5.5) and (5.15) that

$$\overline{\lim_{j\to\infty}} \int_{\mathbb{R}^N} |u_{h_j}(t_0) - u(t_0)|^2 \, dx < \delta,$$

which results in (b) of (5.4) due to the arbitrariness of $\delta > 0$.

[Step 4] The proof of (i), (ii) of Theorem 1.2

By virtue of (5.3), it holds that $\hat{u}_{h_j}$ converges to u everywhere in $[0, \infty) \times \mathbb{R}^N$. As a consequence, passing to the limit as $j \to \infty$ in (5.2) with $h = h_j$, we arrive at (i) and (ii) of Theorem 1.2. □

6. Investigation of convergence as $h \to 0$

In this section we investigate two kinds of convergence properties in terms of the passage $h \to 0$. The first one is $u_h(t_0) \to u(t_0)$ in the space domain $\mathbb{R}^N$ for fixed $t_0 > 0$, and the second one is $u_h \to u$ in time-space domain $Q_T \equiv (0, T) \times \mathbb{R}^N$.

In Definition 1.1, the convergence (1.1) at each time is imposed to hold in the topology induced by the metric of X, which is, in our setting, the one induced by $L^2(\mathbb{R}^N)$-norm as achieved in section 5. On the other hand, restricting our argument to $(u_h) \in \mathrm{DMM}\,(\mathrm{AC}\,; u_0, h)$ constructed by the approximation method proposed in this paper, we can replace it with the topology induced by the uniform norm $||\cdot||_{L^\infty(\mathbb{R}^N)}$.

Proposition 6.1 (Convergence 1). *Let (h_j) be an infinitesimal sequence as in* (5.3) *of the preceding section. Then, for each fixed $t_0 > 0$,*

$$u_{h_j}(t_0) \rightrightarrows u(t_0) \quad \text{in } \mathbb{R}^N \tag{6.1}$$

as $j \to \infty$, where u is the limit function in (5.3).

Proof. Throughout this proof, we fix $t_0 > 0$ arbitrarily. If our conclusion failed, we could find a positive number ε_0 satisfying the following: by taking a subsequence of $(u_{h_j})_{j=1}^\infty$, if necessary, there exists a point $x_j \in \mathbb{R}^N$ for each $j \in \mathbb{N}$ such that

$$|u_{h_j}(t_0, x_j) - u(t_0, x_j)| \geq \varepsilon_0. \tag{6.2}$$

Let n_0 be a natural number such that $t_0 \in (t_{n_0-1}, t_{n_0}]$. Then, for any $x, x' \in \mathbb{R}^N$,

$$\begin{aligned} |u_{h_j}(t_0, x) - u_{h_j}(t_0, x')| &= |u_{n_0,h_j}(x) - u_{n_0,h_j}(x')| = \\ &= |\hat{u}_{h_j}(t_{n_0}, x) - \hat{u}_{h_j}(t_{n_0}, x')| \leq L|x - x'| \end{aligned}$$

by the second inequality of (5.2) with $h = h_j$. In addition, due to Theorem 1.2, we also have the same type of inequality for u:

$$|u(t_0, x) - u(t_0, x')| \leq L|x - x'|.$$

The last two inequalities imply that the function $x \in \mathbb{R}^N \mapsto (u_{h_j}(t_0, x) - u(t_0, x))$ is Lipschitz continuous on $\mathbb{R}^N$ with Lipschitz coefficient $2L$. Accordingly, we find from (6.2)

$$|u_{h_j}(t_0) - u(t_0)| \geq \frac{\varepsilon_0}{2} \quad \text{in } B_{\frac{\varepsilon_0}{4L}}(x_j). \tag{6.3}$$

Therefore

$$||u_{h_j}(t_0) - u(t_0)||_{L^2(\mathbb{R}^N)} \geq \frac{\varepsilon_0}{2}\Big(\frac{\varepsilon_0}{4L}\Big)^{\frac{N}{2}} \mathcal{L}^N(B_1(0))^{\frac{1}{2}}$$

for $j \in \mathbb{N}$, which gives an obvious contradiction to (b) of (5.4). $\square$

We turn our attention to the convergence $u_h \to u$ in time-space domain.

Proposition 6.2 (Convergence 2). *Let (h_j) be an infinitesimal sequence as in (5.3) of the preceding section. For any $T > 0$,*

(i) (a) $u_{h_j} \to u$ *in* $L^2(Q_T)$, (b) $\hat{u}_{h_j} \to u$ *in* $L^2(Q_T)$.

(ii) (a) $u_{h_j} \rightrightarrows u$ *in* Q_T, (b) $\hat{u}_{h_j} \rightrightarrows u$ *in* Q_T

as $j \to \infty$, where $Q_T = (0, T) \times \mathbb{R}^N$.

Before showing Proposition 6.2, we shall first show Lemma 6.3 stated below. We notice that Lemma 6.3 applies to any MM, namely, Lemma 6.3 is demonstrated without restricting our argument to MM constructed in the way proposed by this paper.

Lemma 6.3 (Energy inequality for MM). *Let $u(t)$ be an arbitrary element of* $\mathrm{MM}\,(\mathrm{AC}\,; u_0)$. *Then it holds*

$$\mathrm{AC}\,(u(t)) \leq \mathrm{AC}\,(u_0) \quad \textit{for any } t \geq 0.$$

Proof. Let t_0 be an arbitrarily fixed nonnegative number. By the definition of MM there exists an infinitesimal sequence (h_j) and for each $j \in \mathbb{N}$ an element $(u_{n,h_j})_{n=0}^{\infty} \in \mathrm{DMM}\,(\mathrm{AC}\,; u_0, h)$ such that

$$u_{h_j}(t) \to u(t) \quad \text{in } L^2(\mathbb{R}^N) \tag{6.4}$$

as $j \to \infty$ for each $t \geq 0$, where $u_{h_j}(t)$ is the piecewise constant function generated by the sequence (u_{n,h_j}). Due to the minimality of u_{n,h_j} with respect to the energy functional $\mathrm{AC}_{\,n,h_j}$, taking u_{n-1,h_j} as an energy comparison function we get

$$\begin{aligned}&\int_{\mathbb{R}^N} \Big(\frac{|u_{n,h_j} - u_{n-1,h_j}|^2}{h} + |\nabla u_{n,h_j}|^2 + \chi(u_{n,h_j}) \Big)\, dx \leq \\ &\leq \int_{\mathbb{R}^N} \Big(|\nabla u_{n-1,h_j}|^2 + \chi(u_{n-1,h_j}) \Big)\, dx.\end{aligned}$$

By dropping the fractional term on the left-hand side, we obtain $\mathrm{AC}\,(u_{n,h_j}) \leq \mathrm{AC}\,(u_{n-1,h_j})$ for $n \in \mathbb{N}$. Applying inductively this inequality, we observe that $\mathrm{AC}\,(u_{n,h_j}) \leq \mathrm{AC}\,(u_0)$. Hence, we find $\mathrm{AC}\,(u_h(t_0)) \leq \mathrm{AC}\,(u_0)$, that is,

$$\int_{\mathbb{R}^N} \Big(|\nabla u_{h_j}(t_0)|^2 + \chi(u_{h_j}(t_0))\Big)\, dx \leq \mathrm{AC}\,(u_0). \tag{6.5}$$

Owing to this boundedness and (6.4), we can extract a subsequence, still denoted by the same notation (u_{h_j}), such that

$$u_{h_j}(t_0) \to u(t_0) \quad \text{a.e. in } \mathbb{R}^N, \tag{6.6}$$

$$\nabla u_{h_j}(t_0) \to \nabla u(t_0) \quad \text{weakly in } L^2(\mathbb{R}^N) \tag{6.7}$$

as $j \to \infty$. By employing the Banach-Steinhaus theorem, it follows from (6.7) that

$$\int_{\mathbb{R}^N} |\nabla u(t_0)|^2\, dx \leq \varliminf_{j\to\infty} \int_{\mathbb{R}^N} |\nabla u_{h_j}(t_0)|^2\, dx. \tag{6.8}$$

Furthermore, by (6.6), with the help of the Lebesgue convergence theorem we discover that for any $R > 0$

$$\int_{B_R(0)(u(t_0)>0)} \chi(u(t_0))\, dx = \lim_{j\to\infty} \int_{B_R(0)(u(t_0)>0)} \chi(u_{h_j}(t_0))\, dx.$$

Hence,

$$\int_{B_R(0)} \chi(u(t_0))\, dx \leq \varliminf_{j\to\infty} \int_{\mathbb{R}^N} \chi(u_{h_j}(t_0))\, dx.$$

Passing to the limit as $R \uparrow \infty$, we see from the monotone convergence theorem that

$$\int_{\mathbb{R}^N} \chi(u(t_0))\, dx \leq \varliminf_{j\to\infty} \int_{\mathbb{R}^N} \chi(u_{h_j}(t_0))\, dx. \tag{6.9}$$

By means of (6.5), we conclude from (6.8) and (6.9) that

$$\begin{aligned} \mathrm{AC}\,(u_0) &\geq \varliminf_{j\to\infty} \int_{\mathbb{R}^N} \Big(|\nabla u_{h_j}(t_0)|^2 + \chi(u_{h_j}(t_0))\Big)\, dx \geq \\ &\geq \int_{\mathbb{R}^N} \Big(|\nabla u(t_0)|^2 + \chi(u(t_0))\Big)\, dx = \mathrm{AC}\,(u(t_0)). \end{aligned}$$

The proof of Proposition 6.2

(i) From (6.5) we, in particular, have

$$\mathcal{L}^N(\{u_{n,h_j} > 0\}) \leq \mathrm{AC}\,(u_0) \quad \text{for } n, j \in \mathbb{N}. \tag{6.10}$$

By a truncation argument based on the minimality of u_{n,h_j} and by induction with respect to n we have

$$0 \le u_{n,h_j} \le \sup_{\mathbb{R}^N} u_0 \quad \text{in } \mathbb{R}^N \quad \text{for } n, j \in \mathbb{N}. \tag{6.11}$$

Along with (6.10) and (6.11) we obtain

$$\sup_{\substack{j\in\mathbb{N}\\ t>0}} \int_{\mathbb{R}^N} |u_{h_j}(t)|^2 \, dx < +\infty. \tag{6.12}$$

On the other hand, by (i) of Theorem 1.2 and Lemma 6.3 we also have the corresponding properties for $u(t)$ as follows:

$$\begin{cases} 0 \le u(t) \le \sup_{\mathbb{R}^N} u_0 \quad \text{in } \mathbb{R}^N, \\ \mathcal{L}^N(\{u(t) > 0\}) \le \mathrm{AC}\,(u_0) \quad n, j \in \mathbb{N}. \end{cases}$$

We thereby discover

$$\sup_{t>0} \int_{\mathbb{R}^N} |u(t)|^2 \, dx < +\infty. \tag{6.13}$$

Thus, from (6.12) and (6.13) we find

$$\sup_{\substack{j\in\mathbb{N}\\ t>0}} \int_{\mathbb{R}^N} |u_{h_j}(t) - u(t)|^2 \, dx < +\infty.$$

Due to this and (b) of (5.4), which follows from our assumption (5.3), we see

$$\int_{Q_T} |u_{h_j} - u|^2 \, dz = \int_0^T dt \int_{\mathbb{R}^N} |u_{h_j}(t) - u(t)|^2 \, dx \to 0 \quad (j \to \infty)$$

with the help of Lebesgue convergence theorem. We have thus accomplished the proof of (a). The convergence (b) follows from (a). Indeed,

$$||\hat{u}_{h_j} - u||_{L^2(Q_T)} \le ||\hat{u}_{h_j} - u_{h_j}||_{L^2(Q_T)} + ||u_{h_j} - u||_{L^2(Q_T)}. \tag{6.14}$$

Since the last term of the right-hand side converges to 0 because of (a) of (i), we only have to show that the first term also converges to zero. This is proved in the following fashion:

$$
\begin{aligned}
&\|\hat{u}_{h_j} - u_{h_j}\|^2_{L^2(Q_T)} = \\
&= \int_0^T \int_{\mathbb{R}^N} |\hat{u}_{h_j}(t,x) - u_{h_j}(t,x)|^2 \, dtdx \le \\
&\le \sum_{n=1}^{[\frac{T}{h_j}]+1} \int_{\mathbb{R}^N} |\hat{u}_{h_j}(t,x) - u_{h_j}(t,x)|^2 \, dtdx = \\
&= \sum_{n=1}^{[\frac{T}{h_j}]+1} h_j \int_{\mathbb{R}^N} \left| \frac{t_{n+1} - t}{h_j} u_{n+1,h_j} + \frac{t - t_n}{h_j} u_{n,h_j} - u_{n+1,h_j} \right|^2 dx \le \\
&\le \sum_{n=1}^{[\frac{T}{h_j}]+1} h_j \int_{\mathbb{R}^N} |u_{n,h_j} - u_{n+1,h_j}|^2 \, dx \le \mathrm{AC}\,(u_0) \sum_{n=1}^{[\frac{T}{h_j}]+1} h_j^2,
\end{aligned}
$$

where we use (5.9) with $h = h_j$ for the last inequality. Since the last expression is evaluated from above by the value $h_j(T + h_j)$, the desired result is attained.

Let us turn our attention to the proof of (ii). The assertion (a) immediately follows from (b), because it holds that

$$
|\hat{u}_{h_j}(z) - u_{h_j}(z)| \le H h_j^{\frac{1}{2}} \tag{6.15}
$$

holds for each $j \in \mathbb{N}$ and $z = (t,x) \in (0,\infty) \times \mathbb{R}^N$. This is shown by (5.7) with $h = h_j$ and the second inequality of (5.2) in the following manner:

$$
|\hat{u}_{h_j}(z) - u_{h_j}(z)| \le |u_{n,h_j}(x) - u_{n-1,h_j}(x)| = |\hat{u}_{h_j}(t_n^{(h_j)}, x) - \hat{u}_{h_j}(t_{n-1}^{(h_j)}, x)| \le H h_j^{\frac{1}{2}}.
$$

Here n is a natural number such that $t \in (t_{n-1}^{(h_j)}, t_n^{(h_j)}]$.

The proof of (b) is done by the contradiction argument similar to the proof of Proposition 6.1. More precisely, we use not only the uniform Lipschitz continuity with respect to the space variable but also the uniform Hölder continuity (5.2) and the second inequality of (ii) of Theorem 1.2. As a result, we can arrive at a fact which contradicts (b) of (i). □

References

[1] H.W. Alt – L.A.Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math., **325** (1981), pp. 105-144.

[2] H.W. Alt – L.A.Caffarelli – A.Friedman, *A free boundary problem for quasi-Linear elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., (4), **11** (1984), pp. 1-44.

[3] L. Ambrosio, *Movimenti minimizzanti*, Rend. Accad. Naz. Sci. dei XL, Mem. Mat., **113** (1995), pp. 191-246.

[4] L. Ambrosio – A. Braides, *Energies in SBV and variational models in fracture mechanics*, Homogenization and applications to material sciences (Nice, 1995), GAKUTO Internat. Ser. Math. Sci. Appl., **9**, Gakkōtosho, Tokyo, 1995. Oxford Science Publication (2000), pp. 1-22,

[5] L. Ambrosio – N.Gigli – G.Savaré, *Gradient Flows in metric spaces and in the spaces of probability measures*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag. Basel, 2005.

[6] L. Ambrosio – V.M. Tortorelli, *On the Approximation of Free Discontinuity Problems*, Bollettino U.M.I., (7), **6-B** (1992), pp.105-123.

[7] L. Ambrosio – V.M.Tortorelli, *Approximation of Functionals Depending on Jumps by Elliptic Functionals via Γ-Convergence*, Comm. Pure Appl. Math., Vol. **XLIII** (1990), pp. 999-1036.

[8] A. Braides, *Γ-Convergence for beginners*, Oxford, 2002.

[9] S. Campanato, Equazioni ellittiche del secondo ordine e spazi $\mathcal{L}^{2,\lambda}$. Ann. Mat. Pura Appl. **69** (1965), pp. 321–382.

[10] A. Chambolle, – F. Doveri *Minimizing Movements of the Mumford and Shah energy*, Discrete and continuous Dynamical Systems, Vol **3** (1997), No 2, April, pp. 153-174.

[11] L.A. Caffarelli – J.L. Vázquez *A free-boundary problem for the heat equation arising in flame propagation*, Transaction of the AMS Vol **347** (1995), pp. 411-441.

[12] G. Dal Maso, *An Introduction to Γ-Convergence*, Birkhäuser, 1993.

[13] G. Da Prato, Spazi $\mathcal{L}^{(p,\theta)}(\Omega, \delta)$ e loro proprietà. Ann. Mat. Pura Appl., **69** (1965), pp. 383–392.

[14] M. Gobbino, *Gradient Flow for the One-Dimensional Mumford-Shah Functional*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **27** (1998), pp.145-193.

[15] K. Ito – H. Kazama – K. Nakagawa – K. Svadlenka, *Numerical solution of a volume-constrained free boundary problem by the discrete Morse flow method*, Gakuto International Series, Mathematical Sciences and Application, Vol. **29** (2008), pp. 383-398.

[16] N. Kato – Y. Yamaura, *Uniform Hölder continuity of approximate solutions to parabolic systems and its application*, To be submitted.

[17] N. Kikuchi, – M. Misawa, *Campanato type estimates for solutions of difference-elliptic partial differential equations with constant coefficients*, Rend. Circ. Mat. Palermo, (2) **39**, no.1 (1990), pp. 99–131.

[18] N. Kato – N. Kikuchi, *Campanato-type boundary estimates for Rothe's scheme to parabolic partial differential systems with constant coefficients.* Discrete Contin. Dyn. Syst., **19** (4) (2007), pp. 737–760.

[19] N. Kikuchi, *A construction method of Morse flows to variational functionals*, Nonlinear World, **1** (1994), pp. 131–147.

[20] A. Marino – C. Saccon – M. Tosques, *Curves of maximal slope and parabolic variational inequalities on non-convex constraints.* Ann. Scuola Norm Sup. Pisa Cl. Sci. (4), **16**, (2), (1989) pp. 281-330.

[21] T. Nagasawa, – S. Omata, *Discrete Morse semiflows of a functional with free boundary*, Advances in Mathematical Sciences and Applications, Gakkōtosho, Tokyo, Vol **2**, No 1 (1993) pp. 147–187.

[22] S. Omata, – K. Švadlenka, *Mathematical analysis of a constrained parabolic free boundary problem describing droplet motion on a surface.* Indiana Univ. Math. J., **58** (5) (2009) pp. 2073–2102.

[23] C. Saccon, *On an evolution problem with free boundary*, Houston Journal of Mathematics, Vol **16**, No.2 (1990), pp. 87–120.

[24] Y. Yamaura, *A construction of a Lipschitz continuous minimizer of a free boundary problem*, Nonlinear Analysis **54** (2003), pp. 1175–1191.

[25] Y, Yamaura, *A Gradient Flow of the One-Dimensional Alt-Caffarelli Functional*, Mediterr. J. of Math. no. **4** (2007), pp. 451– 482.

[26] Y. Yamaura, *Hölder-continuity of discrete Morse flows to the Alt-Caffarelli variational problem generating free boundaries*, Nonlinear Analysis, **74** (2011), pp. 7168–7183.

Received submission date; revised revision date