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On the $\mathcal{F}^*$norm of a finite group

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Abstract – Let G be a finite group and $\mathcal{F}$ be a non-empty formation. We define the $\mathcal{F}^*$-norm, denoted by $N^*_{\mathcal{F}}(G)$, to be intersection of the normalizers of the $\mathcal{F}$-residuals of all F-subgroups of G, where $F = \mathcal{N}\mathcal{F}$ is the class of all groups whose $\mathcal{F}$-residuals are nilpotent. In this paper, we research the properties of $N^*_{\mathcal{F}}(G)$ and investigate the relationship between $N^*_{\mathcal{F}}(G)$ and $N_{\mathcal{F}}(G)$, where $N_{\mathcal{F}}(G)$ is the intersection of the normalizers of the $\mathcal{F}$-residuals of all subgroups of G. We show that $N^*_{\mathcal{F}}(G) = N_{\mathcal{F}}(G)$ if $\mathcal{A} \subseteq \mathcal{F} \subseteq \mathcal{N}$.

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1. Introduction

All groups considered in this paper are finite and G denotes a finite group. Our notation and terminology are standard, as in [4]. A formation is a class of groups which is closed under taking epimorphic images and subdirect products. We use $\mathcal{A}$, $\mathcal{N}$ and $\mathcal{U}$ to denote the class of all abelian groups, the class of all nilpotent groups and the class of all supersolvable groups, respectively. The $\mathcal{F}$-residual of G, denoted by $G^{\mathcal{F}}$, is the smallest normal subgroup N such that $G/N \in \mathcal{F}$. In addition, $F = \mathcal{N}\mathcal{F}$ denotes the class of finite groups G with $G^{\mathcal{F}}$ nilpotent.

By $D(G)$ denote the intersection of the normalizers of all derived subgroups of G and by $S(G)$ denote the intersection of the normalizers of the nilpotent residuals of all subgroups of G. Those concepts were introduced in [5] and [7], respectively. Subsequently, Su and Wang [9] introduced the

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concept of $N_{\mathcal{F}}(G)$, the intersection of the normalizers of the $\mathcal{F}$-residuals of all subgroups of G.

Definition 1.1. Let $\mathcal{F}$ be a non-empty formation. By $N_{\mathcal{F}}(G)$ denote the intersection of the normalizers of the $\mathcal{F}$-residuals of all subgroups of G. That is

$$N_{\mathcal{F}}(G) = \bigcap_{H \leq G} N_G(H^{\mathcal{F}}),$$

where $H^{\mathcal{F}}$ is the $\mathcal{F}$-residual of H.

Set $N_{\mathcal{F}}^0(G) = 1$ and if $N_{\mathcal{F}}^i(G)$ is defined, set $N_{\mathcal{F}}^{i+1}(G)/N_{\mathcal{F}}^i(G) = N_{\mathcal{F}}(G/N_{\mathcal{F}}^i(G))$. Let $N_{\mathcal{F}}^{\infty}(G) = N_{\mathcal{F}}^k(G)$ for some integer k such that $N_{\mathcal{F}}^k(G) = N_{\mathcal{F}}^{k+1}(G)$.

Inspired by this concept, we give the following definition:

Definition 1.2. Let the $\mathcal{F}^*$-norm, $N_{\mathcal{F}}^*(G)$, be defined as follows:

$$N_{\mathcal{F}}^*(G) = \bigcap_{H \in T(G)} N_G(H^{\mathcal{F}}),$$

where $H^{\mathcal{F}}$ is the $\mathcal{F}$-residual of H and $T(G) = \{H : H \leq G, H^{\mathcal{F}} \in \mathcal{N}\}$.

Set $N_{\mathcal{F}}^{*0}(G) = 1$ and if $N_{\mathcal{F}}^{*i}(G)$ is defined, set $N_{\mathcal{F}}^{*(i+1)}(G)/N_{\mathcal{F}}^{*i}(G) = N_{\mathcal{F}}^*(G/N_{\mathcal{F}}^{*i}(G))$. Let $N_{\mathcal{F}}^{*\infty}(G) = N_{\mathcal{F}}^{*n}(G)$ for some integer n such that $N_{\mathcal{F}}^{*n}(G) = N_{\mathcal{F}}^{*(n+1)}(G)$.

In this paper, we first show some basic properties about the subgroups $N_{\mathcal{F}}^*(G)$ and $N_{\mathcal{F}}^{*\infty}(G)$. On the other hand, we note a result proved by Gong, Isaacs, Ballester-Bolinches, Kamornikov, and Meng in [1, 2], that is, if a subgroup M of G normalizes $H^{\mathcal{F}}$ for every non-subnormal subgroup H of G, then M normalizes $H^{\mathcal{F}}$ for all subgroups H of G. Inspired by this idea and the fact $N_{\mathcal{F}}(G) \leq N_{\mathcal{F}}^*(G)$, naturally, we may ask a question: Under what conditions does $N_{\mathcal{F}}^*(G) = N_{\mathcal{F}}(G)$ hold?

Remark 1.3. We will explain why we choose the formation $\mathcal{N}$ in $T(G) = \{H : H \leq G, H^{\mathcal{F}} \in \mathcal{N}\}$. Consider the group $G = A_5$ and let $\mathcal{F} = \mathcal{U}$. Suppose that $T(G) = \{H : H \leq G, H^{\mathcal{F}} \in \mathcal{C}\}$, where $\mathcal{C}$ is the class of all cyclic groups. In this case, $G = N_{\mathcal{F}}^*(G) = A_5$ but $N_{\mathcal{F}}(G) = 1$, which indicates that $N_{\mathcal{F}}(G) < N_{\mathcal{F}}^*(G)$. We may note that if we choose the class $\mathcal{C}$ in $T(G)$, then $N_{\mathcal{F}}^*(G)$ will be close to G. It is unsuitable for the question we want to study. And some results below also illustrate that the selection of the formation $\mathcal{N}$ in $T(G)$ is reasonable.

2. Preliminaries

In this section, we list some lemmas and prove some basic properties about the subgroups $N^*_{\mathcal{F}}(G)$ and $N^{*\infty}_{\mathcal{F}}(G)$.

Lemma 2.1 ([4], p.270, Satz 3.5). *Let N and D be two normal subgroups of G satisfying that $D \leq N$ and $D \leq \Phi(G)$. If N/D is nilpotent, then N itself is nilpotent.*

Lemma 2.2. *$F = \mathcal{N}\mathcal{F}$ is a saturated formation for any non-empty formation $\mathcal{F}$. That is: $G \in F$ whenever $G/\Phi(G) \in F$.*

Proof. Suppose that $G/\Phi(G) \in F$. By definition, $(G/\Phi(G))^{\mathcal{F}} \in \mathcal{N}$. It follows that $G^{\mathcal{F}}/(G^{\mathcal{F}} \cap \Phi(G)) \cong G^{\mathcal{F}}\Phi(G)/\Phi(G) = (G/\Phi(G))^{\mathcal{F}} \in \mathcal{N}$. Therefore $G^{\mathcal{F}} \in \mathcal{N}$ by Lemma 2.1, as wanted. □

Lemma 2.3. *Let $\mathcal{F}$ be a non-empty formation. If $M \leq G$ and $N \trianglelefteq G$, then*
(1) $M \cap N^*_{\mathcal{F}}(G) \leq N^*_{\mathcal{F}}(M)$ and so $M \cap N^{*\infty}_{\mathcal{F}}(G) \leq N^{*\infty}_{\mathcal{F}}(M)$;
(2) $N^*_{\mathcal{F}}(G)N/N \leq N^*_{\mathcal{F}}(G/N)$ and so $N^{*\infty}_{\mathcal{F}}(G)N/N \leq N^{*\infty}_{\mathcal{F}}(G/N)$.

Proof. (1) Clearly, $T(M) \leq T(G)$ whenever $M \leq G$. Then

$$\begin{aligned} M \cap N^*_{\mathcal{F}}(G) = M \bigcap \Big(\bigcap_{H \in T(G)} N_G(H^{\mathcal{F}}) \Big) &\leq M \bigcap \Big(\bigcap_{H \in T(M)} N_G(H^{\mathcal{F}}) \Big) \\ &= \bigcap_{H \in T(M)} N_M(H^{\mathcal{F}}) = N^*_{\mathcal{F}}(M). \end{aligned}$$

By induction, it is clear that $M \cap N^{*\infty}_{\mathcal{F}}(G) \leq N^{*\infty}_{\mathcal{F}}(M)$.

(2) For each $H/N \in T(G/N)$, there exists a subgroup B of H such that $H = BN$ and $B \cap N \leq \Phi(B)$. Then we have $H/N \cong BN/N \cong B/B \cap N \in \mathcal{N}\mathcal{F}$. Since $(B/B \cap N)/(\Phi(B)/B \cap N) \cong B/\Phi(B) \in \mathcal{N}\mathcal{F}$, by Lemma 2.2, we get $B \in T(G)$. By definition, for any elements $x \in N^*_{\mathcal{F}}(G)$, we have $x \in N_G(B^{\mathcal{F}})$. It follows that $xN \in N_{G/N}(B^{\mathcal{F}}N/N)$. On the other hand, $(H/N)^{\mathcal{F}} = H^{\mathcal{F}}N/N = (BN)^{\mathcal{F}}N/N \leq B^{\mathcal{F}}N/N \leq (H/N)^{\mathcal{F}}$ and so $(H/N)^{\mathcal{F}} = B^{\mathcal{F}}N/N$. It follows that $xN \in N_{G/N}(H^{\mathcal{F}}/N)$ and hence $N^*_{\mathcal{F}}(G)N/N \leq N^*_{\mathcal{F}}(G/N)$.

By induction, we can show $N^{*\infty}_{\mathcal{F}}(G)N/N \leq N^{*\infty}_{\mathcal{F}}(G/N)$. □

Lemma 2.4. *Let $\mathcal{F}$ be a non-empty formation. If $N \trianglelefteq G$ and $N \leq N^{*\infty}_{\mathcal{F}}(G)$, then $N^{*\infty}_{\mathcal{F}}(G)/N = N^{*\infty}_{\mathcal{F}}(G/N)$.*

Proof. Since $N \le N_{\mathcal{F}}^{*\infty}(G)$, $N \le N_{\mathcal{F}}^{*i}(G)$ for some i. Let $N_{\mathcal{F}}^{*}(G)_1/N = N_{\mathcal{F}}^{*}(G/N)$ and $N_{\mathcal{F}}^{*}(G)_{\infty}/N$ be the terminal term of the ascending series of G/N. We claim that $N_{\mathcal{F}}^{*}(G)_1 \le N_{\mathcal{F}}^{*(i+1)}(G)$. For any element $x \in N_{\mathcal{F}}^{*}(G)_1$, by definition, x normlizes $(H/N)^{\mathcal{F}} = H^{\mathcal{F}}N/N$ where $H/N \in T(G/N)$, namely, $(H^{\mathcal{F}})^x N/N = H^{\mathcal{F}}N/N$. As $N \le N_{\mathcal{F}}^{*i}(G)$, we have x normlizes $(H/N_{\mathcal{F}}^{*i}(G))^{\mathcal{F}} = H^{\mathcal{F}}N_{\mathcal{F}}^{*i}(G)/N_{\mathcal{F}}^{*i}(G)$ where $H/N_{\mathcal{F}}^{*i}(G) \in T(G/N_{\mathcal{F}}^{*i}(G))$ since $H/N \in T(G/N)$. Hence $x \in N_{\mathcal{F}}^{*(i+1)}(G)$ and so the claim holds. By induction, we have $N_{\mathcal{F}}^{*}(G)_{\infty} \le N_{\mathcal{F}}^{*\infty}(G)$. Conversely, it is easy to see that $N_{\mathcal{F}}^{*}(G) \le N_{\mathcal{F}}^{*}(G)_1$. By induction, we have $N_{\mathcal{F}}^{*\infty}(G) \le N_{\mathcal{F}}^{*}(G)_{\infty}$. Hence $N_{\mathcal{F}}^{*\infty}(G) = N_{\mathcal{F}}^{*}(G)_{\infty}$ and so $N_{\mathcal{F}}^{*\infty}(G)/N = N_{\mathcal{F}}^{*}(G)_{\infty}/N = N_{\mathcal{F}}^{*\infty}(G/N)$. The proof is complete. □

Lemma 2.5. *Let G be a group and $\mathcal{F}$ be a non-empty subgroup-closed formation. Then $C_G(G^{\mathcal{F}}) \le N_{\mathcal{F}}(G)$.*

Proof. Let $H \le G$. Since $\mathcal{F}$ is subgroup-closed and $G/G^{\mathcal{F}} \in \mathcal{F}$, we have $HG^{\mathcal{F}}/G^{\mathcal{F}} \in \mathcal{F}$ and so $(HG^{\mathcal{F}}/G^{\mathcal{F}})^{\mathcal{F}} = H^{\mathcal{F}}G^{\mathcal{F}}/G^{\mathcal{F}} = 1$. Hence $H^{\mathcal{F}} \le G^{\mathcal{F}}$ and therefore $C_G(G^{\mathcal{F}}) \le N_{\mathcal{F}}(G)$. □

Lemma 2.6 ([3], p.34, Lemma 1.7.11). *If H/K is a pd-chief factor of G, then $O_p(G/C_G(H/K)) = 1$.*

3. Main Results

Theorem 3.1. *Let $\mathcal{F}$ be a formation such that $\mathcal{A} \subseteq \mathcal{F} \subseteq \mathcal{U}$. Then the subgroup $N_{\mathcal{F}}^{*\infty}(X)$ is solvable for any group X.*

Proof. Clearly, we only need to prove $N_{\mathcal{F}}^{*}(X)$ is solvable. Write $G = N_{\mathcal{F}}^{*}(X)$. Then $H^{\mathcal{F}}$ is normal in G for every subgroup H in $T(G)$ and so $G = N_{\mathcal{F}}^{*}(G)$. Assume that $G = N_{\mathcal{F}}^{*}(G)$ is not solvable and let G be a counter-example of minimal order.

Let K be a proper subgroup of G. Then by Lemma 2.3(1), $K = K \cap G = K \cap N_{\mathcal{F}}^{*}(G) \le N_{\mathcal{F}}^{*}(K) \le K$ and so $K = N_{\mathcal{F}}^{*}(K)$. By induction, we have that K is solvable. If $N \ne 1$ is a proper normal subgroup of G, then by Lemma 2.3(2), $G/N = N_{\mathcal{F}}^{*}(G)/N \le N_{\mathcal{F}}^{*}(G/N) \le G/N$ and so $N_{\mathcal{F}}^{*}(G/N) = G/N$. By induction, we have that G/N is solvable. Also we can see that N is solvable because $N < G$. Thus G is solvable, a contradiction. It follows that $G = N_{\mathcal{F}}^{*}(G)$ is a minimal simple group.

Since $H^{\mathcal{F}}$ is normal in G for every subgroup H in $T(G)$, we have:

$$H^{\mathcal{F}} = 1, \text{ for all } H \in T(G).$$

On the other hand, all minimal simple groups have been classified by Thompson [10]. There types are:
(i) $PSL(3,3)$;
(ii) The Suzuki group $Sz(2^f)$, where f is an odd prime;
(iii) $PSL(2,p)$, where p is a prime with $p > 3$ and $p^2 + 1 \equiv 0 (mod\ 5)$;
(iv) $PSL(2,2^f)$, where f is a prime;
(v) $PSL(2,3^f)$, where f is an odd prime.

Since each of the groups $PSL(3,3)$, $PSL(2,p)$ and $PSL(2,3^f)$ contains a subgroup H that is isomorphic to A_4, the alternating group of degree 4, we have $H^{\mathcal{F}} \cong K_4$ for $\mathcal{A} \subseteq \mathcal{F} \subseteq \mathcal{U}$ and so H belongs to $T(G)$, which contradicts to $H^{\mathcal{F}} = 1$. Thus, G cannot be any of $PSL(3,3)$, $PSL(2,p)$ and $PSL(2,3^f)$. Suppose that $G \cong PSL(2,2^f)$ or $G \cong Sz(2^f)$. Then G is a Zassenhause group of odd degree and the stabilizer of a point is a Frobenius group with kernel a 2-group. It follows that G cannot be $PSL(2,2^f)$ or $Sz(2^f)$. Therefore the proof is complete. □

Theorem 3.2. *Let $\mathcal{A} \subseteq \mathcal{F} \subseteq \mathcal{N}$. Then $N^*_{\mathcal{F}}(G) = N_{\mathcal{F}}(G)$ for any group G.*

Proof. Clearly, $N_{\mathcal{F}}(G) \leq N^*_{\mathcal{F}}(G)$. Therefore, we only need to prove that $N^*_{\mathcal{F}}(G) \leq N_G(H^{\mathcal{F}})$ for all subgroups H of G.

Assume that this theorem is false and let G be a counter-example of minimal order. Set $K = N^*_{\mathcal{F}}(G)$ and $TK(G) = \{H : H^{\mathcal{F}} \notin \mathcal{N}\ and\ K \nleq N_G(H^{\mathcal{F}})\}$. It is clear that the collection $TK(G)$ is not empty. Let S have the smallest possible order such that $S \in TK(G)$. Moreover, we have:

(1) $G = SK$.

Assume that $SK < G$. By Lemma 2.3(1), we have $K = SK \cap K = SK \cap N^*_{\mathcal{F}}(G) \leq N^*_{\mathcal{F}}(SK)$. Then by the minimal choice of G, we have $N^*_{\mathcal{F}}(SK) \leq N_{SK}(H^{\mathcal{F}})$ for all subgroups H of SK. Hence, $K \leq N_G(S^{\mathcal{F}})$, a contradiction. Thus we conclude that $G = SK$.

(2) If $S \leq Y < G$, then $S^{\mathcal{F}} \trianglelefteq Y$.

By step (1), we have $Y = Y \cap SK = S(Y \cap K)$. By Lemma 2.3(1), $Y \cap K = Y \cap N^*_{\mathcal{F}}(G) \leq N^*_{\mathcal{F}}(Y)$. Then by the minimal choice of G, $N^*_{\mathcal{F}}(Y) \leq N_Y(H^{\mathcal{F}})$ for all subgroups H of Y. Hence $Y \cap K \leq N_Y(S^{\mathcal{F}})$ and so $Y \leq N_Y(S^{\mathcal{F}})$. It follows that $S^{\mathcal{F}} \trianglelefteq Y$, as claimed.

(3) Let $T = (S^{\mathcal{F}})^G$ be the normal closure of $S^{\mathcal{F}}$ in G. Then $T \leq S^{\mathcal{F}}N$ for any non-trivial normal subgroup N of G. Furthermore, $\mathrm{Core}_G(S^{\mathcal{F}}) = 1$.

Consider the factor group KN/N. By Lemma 2.3(2), $KN/N = N^*_{\mathcal{F}}(G)N/N \leq N^*_{\mathcal{F}}(G/N)$. Since $|G/N| < |G|$, $N^*_{\mathcal{F}}(G/N) \leq N_{G/N}((H/N)^{\mathcal{F}})$ for all subgroup H/N of G/N. Hence $KN/N \leq N_{G/N}((SN/N)^{\mathcal{F}}) = N_{G/N}(S^{\mathcal{F}}N/N)$ and so $K \leq N_G(S^{\mathcal{F}}N)$. Since $S \leq N_G(S^{\mathcal{F}}N)$ and by $G = SK$, we have $S^{\mathcal{F}}N \trianglelefteq G$. Therefore, $T \leq S^{\mathcal{F}}N$.

Assume that $\mathrm{Core}_G(S^{\mathcal{F}}) \neq 1$. Then $T = S^{\mathcal{F}}\mathrm{Core}_G(S^{\mathcal{F}}) = S^{\mathcal{F}} \trianglelefteq G$, contrary to $K \nleq N_G(S^{\mathcal{F}})$.

(4) $S = S^{\mathcal{F}}$.

Clearly, $S^{\mathcal{F}} \neq 1$. Let X be a proper subgroup of S. We now claim that X is an $\mathcal{F}$-subgroup of S. If $X^{\mathcal{F}} \in \mathcal{N}$, then by the definition of K, we have $K \leq N_G(X^{\mathcal{F}})$. If $X^{\mathcal{F}} \notin \mathcal{N}$, then by the smallest choice of S, we have $K \leq N_G(X^{\mathcal{F}})$. In both cases, $K \leq N_G(X^{\mathcal{F}})$, and so $(X^{\mathcal{F}})^G = (X^{\mathcal{F}})^{SK} = (X^{\mathcal{F}})^S \leq S^{\mathcal{F}}$. By step (3), $X^{\mathcal{F}} \leq (X^{\mathcal{F}})^G \leq \mathrm{Core}_G(S^{\mathcal{F}}) = 1$, and thus $X \in \mathcal{F}$.

If $S^{\mathcal{F}} < S$, then $S^{\mathcal{F}} \in \mathcal{F}$. Since $\mathcal{A} \subseteq \mathcal{F} \subseteq \mathcal{N}$, we have $S^{\mathcal{F}} \in \mathcal{N}$, contrary to the choice of S. Thus, $S = S^{\mathcal{F}}$, as desired.

(5) Let N be a minimal normal subgroup of G contained in K. Then $G = T = S^{\mathcal{F}}N = SN$.

Because $N \leq K = N^*_{\mathcal{F}}(G)$ is solvable by Theorem 3.1, we have N is an elementary abelian p-subgroup for some prime p. First, we claim that $N \leq T$. If $N \nleq T$, then $N \cap T = 1$. By step (3), $T \leq S^{\mathcal{F}}N$. Hence $T = T \cap S^{\mathcal{F}}N = S^{\mathcal{F}}(T \cap N) = S^{\mathcal{F}}$, a contradiction. Thus $N \leq T$. It follows that $T = S^{\mathcal{F}}N = SN$ by steps (3)(4).

Assume that $T < G$. Since $S \leq T = SN$ and by step (2), $S^{\mathcal{F}} \trianglelefteq T$. Hence $T/S^{\mathcal{F}} \cong N/N \cap S^{\mathcal{F}}$ is an abelian group and so $T' \leq S^{\mathcal{F}}$. By step (3), $T' \leq \mathrm{Core}_G(S^{\mathcal{F}}) = 1$. Therefore T is abelian and so $S^{\mathcal{F}}$ is abelian, which contradicts to the choice of S. Thus $G = T = S^{\mathcal{F}}N = SN$, as desired.

(6) G/C is q-nilpotent for any prime q with $q \neq p$, where $C = C_G(N)$.

Assume that G/C is not q-nilpotent for some fixed prime q with $q \neq p$. Then there exists a non-q-nilpotent subgroup H/C of G/C and all proper subgroups of H/C are q-nilpotent. Choose a subgroup L of Hsuch that $H = CL$ and $C \cap L \leq \Phi(L)$. Then $L/\Phi(L) \cong H/C$, so $L/\Phi(L)$ is a minimal non-q-nilpotent group. According to a theorem of Ito [6, p.296, Theorem 10.3.3], $L/\Phi(L) = Q/\Phi(L) \cdot R/\Phi(L)$ is a minimal non-nilpotent group of order $q^m r^n$, where $Q/\Phi(L)$ is a normal Sylow q-subgroup of $L/\Phi(L)$ and $R/\Phi(L)$ is a cyclic Sylow r-subgroup of $L/\Phi(L)$, $r \neq q$ is a prime, $m, n \geq 1$.

Since $(L/\Phi(L))^{\mathcal{F}}$ is a q-group, $L^{\mathcal{F}}$ is nilpotent by Lemma 2.1 and so $L \in T(G)$. Let Q_0 be a Sylow q-subgroup of $L^{\mathcal{F}}$. Then we have Q_0 char $L^{\mathcal{F}}$. By the definition of $N^*_{\mathcal{F}}(G)$, $N \leq N^*_{\mathcal{F}}(G) \leq N_G(L^{\mathcal{F}})$ and therefore N normalizes Q_0. As Q_0 is a q-subgroup of G with $q \neq p$, we know that $[N, Q_0] = 1$, and so $Q_0 \leq C$. Consequently, $Q_0 \leq C \cap L \leq \Phi(L)$ and hence $L/\Phi(L) \in \mathcal{F}$, which is a contradiction because $\mathcal{A} \subseteq \mathcal{F} \subseteq \mathcal{N}$.

(7) Final contradiction.

By step (5), $G = S^{\mathcal{F}}N = SN$. Furthermore, $S^{\mathcal{F}} \cap N = 1$ and S is a maximal subgroup of G. By step (3)(4), $\mathrm{Core}_G(S^{\mathcal{F}}) = \mathrm{Core}_G(S) = 1$. Hence G is a primitive group with stabilizer S. By [1, p.53, Theorem 15.2], we have $N = C_G(N)$.

By step (6), $G/C = G/N$ is p-closed. By Lemma 2.6, $O_p(G/C) = 1$ which indicates that G/C is a p'-group and is nilpotent. Therefore $G/C = G/N \cong S^{\mathcal{F}}$ is nilpotent, a contradiction. The proof is complete. □

Theorem 3.3. *Let $\mathcal{F}$ be a formation and $F = \mathcal{N}\mathcal{F}$. If $\mathcal{A} \subseteq \mathcal{F} \subseteq \mathcal{U}$ or G is solvable, then the following statements are equivalent:*
(1) $G \in F$;
(2) $G/N^*_{\mathcal{F}}(G) \in F$.

Proof. (1)⇒(2): Clearly.
(2)⇒(1): We use induction on the order of G. The conclusion holds naturally if $N^*_{\mathcal{F}}(G) = 1$. Suppose that $N^*_{\mathcal{F}}(G) > 1$. Then we can find a minimal normal subgroup N of G such that $N \leq N^*_{\mathcal{F}}(G)$. By Theorem 3.1 or the solvability of G, $N^*_{\mathcal{F}}(G)$ is solvable, so N is an elementary abelian p-group for some prime p.

Let $N \leq \Phi(G)$. By Lemma 2.3(2), we have $N^*_{\mathcal{F}}(G)/N \leq N^*_{\mathcal{F}}(G/N)$. It follows that $(G/N)/N^*_{\mathcal{F}}(G/N) \in F$ since $G/N^*_{\mathcal{F}}(G) \in F$. We thus have that G/N satisfies the condition of the theorem. By induction, $G/N \in F$ and so $(G/N)/\Phi(G/N) \in F$. As $N \leq \Phi(G)$, then $\Phi(G)/N = \Phi(G/N)$ and so $(G/N)/\Phi(G/N) = (G/N)/\Phi(G)/N \cong G/\Phi(G) \in F$. Therefore $G \in F$ by Lemma 2.2, as desired.

Let $N \nsubseteq \Phi(G)$. Then there is a maximal subgroup M of G such that $G = NM$ with $N \cap M = 1$. By Lemma 2.3(1), $M \cap N^*_{\mathcal{F}}(G) \leq N^*_{\mathcal{F}}(M)$. Since $G/N^*_{\mathcal{F}}(G) \cong M/M \cap N^*_{\mathcal{F}}(G) \in F$, we have $M/N^*_{\mathcal{F}}(M) \in F$ and so $M \in F$ by induction, which indicates that $M \in T(G)$. As $N \leq N^*_{\mathcal{F}}(G) \leq N_G(M^{\mathcal{F}})$, we have $M^{\mathcal{F}} \trianglelefteq MN = G$ and it follows that $NM^{\mathcal{F}} = N \times M^{\mathcal{F}}$. Since $M^{\mathcal{F}}$

is nilpotent and $G^{\mathcal{F}} \leq N \times M^{\mathcal{F}}$, we conclude that $G^{\mathcal{F}}$ is nilpotent, that is $G \in F$, as desired. □

Theorem 3.4. *Let $\mathcal{F}$ be a subgroup-closed formation and $F = \mathcal{N}\mathcal{F}$. If $\mathcal{A} \subseteq \mathcal{F} \subseteq \mathcal{U}$ or G is solvable, then the following statements are equivalent:*
(1) $G \in F$;
(2) $G = N_{\mathcal{F}}^{\infty}(G)$;
(3) $G = N_{\mathcal{F}}^{*\infty}(G)$.

Proof. (1)⇒(2): It is clear that $G/N_{\mathcal{F}}^{\infty}(G) \in F$. We observe the following fact: If $X > 1$ is an F-group, then $N_{\mathcal{F}}(X) > 1$. In fact, $X^{\mathcal{F}}$ is nilpotent, so $C_X(X^{\mathcal{F}}) > 1$. But $C_X(X^{\mathcal{F}}) \leq N_{\mathcal{F}}(X)$ by Lemma 2.5, we have $N_{\mathcal{F}}(X) > 1$. Using this fact and noting that $N_{\mathcal{F}}(G/N_{\mathcal{F}}^{\infty}(G)) = 1$, we deduce $G = N_{\mathcal{F}}^{\infty}(G)$, the conclusion follows.

(2)⇒(3): Evidently.

(3)⇒(1): Assume that the theorem is false and let G be a counter-example of minimal order. If $N_{\mathcal{F}}^{*}(G) = 1$, then nothing needs to be shown. Suppose that $N_{\mathcal{F}}^{*}(G) > 1$. As $N_{\mathcal{F}}^{*\infty}(G/N_{\mathcal{F}}^{*}(G)) = N_{\mathcal{F}}^{*\infty}(G)/N_{\mathcal{F}}^{*}(G) = G/N_{\mathcal{F}}^{*}(G)$ by Lemma 2.4, then $G/N_{\mathcal{F}}^{*}(G) \in F$ by the choice of G. By Theorem 3.3, we have that $G \in F$. The proof is complete. □

Theorem 3.5. *Let $\mathcal{F}$ be a subgroup-closed formation such that $\mathcal{A} \subseteq \mathcal{F} \subseteq \mathcal{U}$ and $F = \mathcal{N}\mathcal{F}$. Then $N_{\mathcal{F}}^{*\infty}(G) \leq Int_F(G)$, where $Int_F(G)$ is the intersection of all maximal subgroups of G.*

Proof. Firstly, we prove a more general conclusion: if H is an F-subgroup of G, then $N_{\mathcal{F}}^{*\infty}(G)H$ is an F-group. Let $M = N_{\mathcal{F}}^{*\infty}(G)H$, then $N_{\mathcal{F}}^{*}(G) \leq N_{\mathcal{F}}^{*}(M)$ by Lemma 2.3(1). By Lemma 2.3(2) and induction, we have $N_{\mathcal{F}}^{*i}(G) \leq N_{\mathcal{F}}^{*i}(M)$ for all $i \geq 1$ and so $N_{\mathcal{F}}^{*\infty}(G) \leq N_{\mathcal{F}}^{*\infty}(M)$. As $M/N_{\mathcal{F}}^{*\infty}(M) = N_{\mathcal{F}}^{*\infty}(M)H/N_{\mathcal{F}}^{*\infty}(M) \cong H/H \cap N_{\mathcal{F}}^{*\infty}(M) \in F$, it is clear that $M = N_{\mathcal{F}}^{*\infty}(M)$ and so $M \in F$ by Theorem 3.4.

Let L be a maximal F-group of G such that $H \leq L$. Then $N_{\mathcal{F}}^{*\infty}(G)H \leq L$, and thus $N_{\mathcal{F}}^{*\infty}(G) \leq Int_F(G)$, where $Int_F(G)$ is the intersection of all maximal subgroups of G. The proof is complete. □

In [8], Skiba introduced the concept of the boundary condition and Su [9] considered the relationship between $N_{\mathcal{F}}^{\infty}(G)$ and $Int_F(G)$. According to Theorem 3.5 and [9, Theorem C], we have:

Corollary 3.6. *Let $\mathcal{F}$ be a subgroup-closed formation such that $\mathcal{A} \subseteq \mathcal{F} \subseteq \mathcal{U}$ and $F = \mathcal{N}\mathcal{F}$ satisfy the boundary condition. Then $N_{\mathcal{F}}^{\infty}(G) = N_{\mathcal{F}}^{*\infty}(G) = Int_F(G)$.*

Theorem 3.7. *Let $\mathcal{F}$ be a non-empty subgroup-closed formation and $F = \mathcal{N}\mathcal{F}$. If G is a primitive solvable group with stabilizer M satisfying $M \in F$, then*
(1) $N_{\mathcal{F}}^{*}(G) = N_{\mathcal{F}}(G) = C_G(G^{\mathcal{F}}) = 1$ or
(2) $N_{\mathcal{F}}^{*}(G) = N_{\mathcal{F}}(G)$ and $G = N_{\mathcal{F}}^{*\infty}(G) = N_{\mathcal{F}}^{\infty}(G)$.

Proof. Clearly, we have $C_G(G^{\mathcal{F}}) \leq N_{\mathcal{F}}(G) \leq N_{\mathcal{F}}^{*}(G)$ by Lemma 2.5. Since G is a primitive solvable group with stabilizer M, by [1, p.54, Theorem 15.6], G has an unique minimal normal subgroup N, $G = N \rtimes M$ and $N = C_G(N)$ is abelian. If $N_{\mathcal{F}}^{*}(G) = 1$, then $N_{\mathcal{F}}^{*}(G) = N_{\mathcal{F}}(G) = C_G(G^{\mathcal{F}}) = 1$. Hence (1) holds.

Assume that $N_{\mathcal{F}}^{*}(G) \neq 1$. Then $N \leq N_{\mathcal{F}}^{*}(G) \leq N_G(M^{\mathcal{F}})$ due to $M \in F$. It is clear that $M^{\mathcal{F}} \trianglelefteq N \rtimes M = G$ and so $M^{\mathcal{F}} \leq M_G = 1$. Since $G/N \cong M \in \mathcal{F}$, we have $G^{\mathcal{F}} \leq N$. Hence $G^{\mathcal{F}} \in \mathcal{N}$, that is $G \in F$. It follows by the definition of $N_{\mathcal{F}}^{*}(G)$ that $N_{\mathcal{F}}^{*}(G) = N_{\mathcal{F}}(G)$. Furthermore, by Theorem 3.4 we have $G = N_{\mathcal{F}}^{*\infty}(G) = N_{\mathcal{F}}^{\infty}(G)$. Hence (2) holds. □

Example 3.8. Let $G = S_4 = K_4 \rtimes S_3$. It is clear that G satisfies the condition of Theorem 3.7. For $\mathcal{F} = \mathcal{N}$, we have $N_{\mathcal{F}}^{*}(G) = N_{\mathcal{F}}(G) = C_G(G^{\mathcal{F}}) = 1$. For $\mathcal{F} = \mathcal{U}$, we have $N_{\mathcal{F}}^{*}(G) = N_{\mathcal{F}}(G) = S_4$, but meanwhile $C_G(G^{\mathcal{U}}) = C_G(K_4) = K_4 \neq N_{\mathcal{F}}^{*}(G)$.

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