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On generalized Π-property of subgroups of finite groups

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Abstract – In this note, we extend the concept of Π-property of subgroups of finite groups and generalize some recent results. In particular, we generalize the main results of Li et al. [p-Hypercyclically embedding and Π-property of subgroups of finite groups, Comm. Algebra 45(8)(2017), 3468–3474] and Miao et al. [On the supersoluble hypercentre of a finite group, Monatsh. Math. (2016)].

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1. Introduction

Suppose that G is a finite group and p is a prime. Let $\pi(G)$ be the set of all the prime divisors of $|G|$. Let $O^p(G) = \bigcap\{N \mid N \trianglelefteq G$ and G/N is a p-group$\}$. To state our results, we need to recall some notation. According to Kegel (see [7]), let H be a subgroup of a finite group G; then H is called an S-permutable subgroup of G if H permutes with every Sylow subgroup of G. According to Chen (see [2]), let H be a subgroup of a finite group G; then H is said to be S-semipermutable in G if $HQ = QH$ for all Sylow q-subgroups Q of G for all primes q not dividing $|H|$. Recently, in [8], Li introduced the concept of Π-property and Π-normality of subgroups of finite groups. Let H be a subgroup of a finite group G. We call that H satisfies Π-property in G if, for any chief factor K/L of G, $[G/L : N_{G/L}((H \cap K)L/L]$ is a $\pi((H \cap K)L/L)$-number; we call that H is Π-normal in G if there exist a subnormal subgroup T of G and a subgroup I of G satisfying Π-property in G such that $G = HT$ and $H \cap T \leq I \leq H$. It is not very difficult to prove that an S-semipermutable p-subgroup of a finite group G satisfies Π-property in G (see Lemma 2.9).

Following Berkovich and Isaacs (see [1]), if G is a finite group and p is a prime

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divisor of $|G|$, we write G_p^* to denote the unique smallest normal subgroup of G for which the corresponding factor group is abelian of exponent dividing $p-1$. It is well known that G is p-supersolvable if and only if G_p^* is p-nilpotent (see Lemma 3.6 of [1]).

In 2014, Berkovich and Isaacs proved the following theorem.

Theorem 1.1 (Berkovich and Isaacs). *Let p be a prime dividing the order of a finite group G and $P \in Syl_p(G)$.*

(a)(*Lemma* 3.8, [1]) *If P is cyclic and some nonidentity subgroup $U \leq P$ is S-semipermutable in G, then G is p-supersolvable.*

(b)(*Theorem D,* [1]) *Fix an integer $e \geq 3$. If P is a noncyclic p-group with $|P| \geq p^{e+1}$ and every noncyclic subgroup of P with order p^e is S-semipermutable in G, then G is p-supersolvable.*

(c)(*Corollary E,* [1]) *If P is a noncyclic p-group with $|P| \geq p^3$ and every subgroup of P with order p^2 is S-semipermutable in G, then G is p-supersolvable.*

In 2017, Li and Miao proved the following theorem (see [9]).

Theorem 1.2. *Let G be a finite group, M a normal subgroup of G, p a prime divisor of $|M|$, X a normal subgroup of G with $F_p^*(M) \leq X \leq M$ and $P \in Syl_p(M)$. Then every p-chief factor of G below M is cyclic if and only if P has a subgroup D such that $1 < |D| \leq max\{p, |P|/p\}$ and for any subgroup H of P with order $|D|$ (if P is a non-abelian 2-group and $|D| = 2$, also for any cyclic subgroup H of P with order 4), $H \cap O^p(G)$ satisfies Π-property in G.*

Here, as usual, $F_p^*(M)$ is the generalized p-Fitting subgroup of M, i.e., $F_p^*(M)$ is the normal subgroup of M such that $O_{p'}(M) \leq F_p^*(M)$ and $F_p^*(M)/O_{p'}(M) = F^*(M/O_{p'}(M))$ (see [12]).

In this note, we extend the concept of Π-property and Π-normality of subgroups of finite groups and generalize the above results. At first, we introduce the following definition.

Definition 1.3. Let p be a prime dividing the order of a finite group G and $M \trianglelefteq G$. Let $M_G^{*p} = \bigcap\{N \leq M$ and $N \trianglelefteq G \mid$ every p-chief factor of G/N below M/N is cyclic$\}$. It is not very difficult to see that every p-chief factor of G/M_G^{*p} below M/M_G^{*p} is cyclic. And we have $M_M^{*p} \leq M_G^{*p} \leq M \cap G_G^{*p}$.

It is not very difficult to prove that $M_G^{*p} = O^{p'}([M_p^*, O^p(G_p^*)]O^p(M_p^*))$.

In particular, if M is a p-subgroup, then $M_G^{*p} = [M, O^p(G_p^*)]$.

Example 1.4. Let $G = A_4$ and M be the Sylow 2-subgroup of G. It is not very difficult to see that $M_M^{*2} = 1$ and $M_G^{*2} = M$. Then $M_M^{*2} < M_G^{*2}$.

Example 1.5. Let $G = Q_8 \rtimes \mathbb{Z}_3$ and M be the unique subgroup of G with order 2. It is not very difficult to see that $M_G^{*2} = 1$ and $G_G^{*2} = Q_8$. Then $M_G^{*2} = 1 < M = M \cap G_G^{*2}$.

Now we introduce the following definition.

Definition 1.6. Let G be a finite group, $M \trianglelefteq G$ and $H \leq G$. If for any chief factor K/L of G below M, we have $[G/L : N_{G/L}((H\cap K)L/L)]$ is a $\pi((H\cap K)L/L)$-number, then we call that H satisfies Π-property in G with respect to M. Let $\prod_M(G) = \{H \leq G \mid H$ satisfies Π-property in G with respect to $M\}$.

It is not very difficult to prove that H satisfies Π-property in G with respect to M if and only if $H \cap M$ satisfies Π-property in G.

Remark 1.7. Let $N \leq M$ be normal subgroups of a finite group G. It is not very difficult to see that $\prod_M(G) \subseteq \prod_N(G)$.

Remark 1.8. There exists a finite group G with p is a prime divisor of $|G|$ such that G has a p-subgroup P_1 with $P_1 \in \prod_{G_G^{*p}}(G)$, but $P_1 \notin \prod_{O^p(G)}(G)$. See the following example.

Example 1.9. Let $p = 5$ and $G = \langle a, b, d \mid a^5 = b^5 = d^3 = 1, [a, b] = 1, d^{-1}ad = b, d^{-1}bd = a^{-1}b^{-1}\rangle \times \langle c, f \mid c^5 = f^2 = 1, f^{-1}cf = c^{-1}\rangle \cong ((\mathbb{Z}_5 \times \mathbb{Z}_5) \rtimes \mathbb{Z}_3) \times D_{10}$. By Fitting's Theorem (see Theorem 4.34 of [5]), it follows that $G_G^{*p} = \langle a\rangle \times \langle b\rangle$ and $O^p(G) = G$. Let $P_1 = \langle ac\rangle$. Then $P_1 \cap G_G^{*p} = 1$, and thus $P_1 \in \prod_{G_G^{*p}}(G)$. Since $\langle a\rangle \ntrianglelefteq G$, it follows that $P_1 \notin \prod_{O^p(G)}(G)$.

Remark 1.10. There exists a finite group G with $M \trianglelefteq G$ and p is a prime divisor of $|M|$ such that M has a p-subgroup P_1 with $P_1 \in \prod_{M_G^{*p}}(G)$, but $P_1 \notin \prod_{G_G^{*p}}(G)$. See the following example.

Example 1.11. Let $p = 5$. Consider $P = \langle a, b, c \mid a^5 = b^5 = c^5 = 1,\ [a, b] = [a, c] = 1,\ c^{-1}bc = ab\rangle$. Then $|P| = p^3$ and $\Phi(P) = \langle a\rangle$. There exists $d \in Aut(P)$ such that $a^d = a, b^d = c^{-1}b^{-1}$ and $c^d = ab$. In $Aut(P)$, we have $\circ(d) = 3$. Consider the semidirect product $G_1 = P \rtimes \langle d\rangle$. Consider $G_2 = \langle f, g, h \mid f^5 = g^5 = h^3 = 1, [f, g] = 1, h^{-1}fh = g, h^{-1}gh = f^{-1}g^{-1}\rangle$. Let $G = G_1 \times G_2$, $M = \langle a\rangle \times \langle f\rangle \times \langle g\rangle$ and $P_1 = \langle af\rangle$. It is not very difficult to see that $M \trianglelefteq G$. Note that $G_p^* = G$. By Fitting's Theorem, it is not very difficult to prove that $O^p(G_p^*) = G$. Hence $G_G^{*p} = P \times \langle f\rangle \times \langle g\rangle$. It is not very difficult to see that $M_G^{*p} = \langle f\rangle \times \langle g\rangle$. Since $P_1 \cap M_G^{*p} = 1$, it follows that $P_1 \in \prod_{M_G^{*p}}(G)$. Since $\langle f\rangle \ntrianglelefteq G$, we see that $P_1 \notin \prod_{G_G^{*p}}(G)$.

Let p be a prime and P be a nonidentity p-group with $|P| = p^n$. We define the set $\mathbb{L}_1(P)$. If $p = 2$ and P is non-abelian, let $\mathbb{L}_1(P) = \{P_1 \mid P_1 \leq P$ and $|P_1| = 2\} \cup \{P_2 \mid P_2 \leq P$ and P_2 is a cyclic subgroup of order $4\}$. Otherwise, let $\mathbb{L}_1(P) = \{P_1 \mid P_1 \leq P$ and $|P_1| = p\}$.

In this note, we prove the following result.

Theorem 1.12. *Let G be a finite group, $M \trianglelefteq G$, p be a prime divisor of $|M|$, $e \geq 2$ be an integer, and $P \in Syl_p(M)$ with $|P| \geq p^{e+1}$ and P is noncyclic. Suppose* *that for any normal noncyclic subgroup P_1 of P with order p^e (if P has such a subgroup), $P_1 \in \prod_{M_G^{*p}}(G)$. If $|P \cap M_G^{*p}| \le p^e$ or $P \cap M_G^{*p}$ is cyclic, then every p-chief factor of G below M is cyclic.*

By Theorem 1.12, we obtain the following results.

Theorem 1.13. *Let G be a finite group and $X \le M$ be normal subgroups of G with $F_2^*(M) \le X \le M$. Suppose that X_G^{*2} has a cyclic Sylow 2-subgroup. Then every chief factor of $G/O_{2'}(M)$ below $M/O_{2'}(M)$ is cyclic. In particular, every 2-chief factor of G below M is cyclic.*

Theorem 1.14. *Let G be a finite group, $X \le M$ be normal subgroups of G with $p > 2$ is a prime divisor of $|M|$ and $F_p^*(M) \le X \le M$, and $P \in Syl_p(X)$. Suppose that P is cyclic and there exists $1 < P_1 \le P$ such that $P_1 \in \prod_{X_G^{*p}}(G)$. Then every chief factor of $G/O_{p'}(M)$ below $M/O_{p'}(M)$ is cyclic. In particular, every p-chief factor of G below M is cyclic.*

Theorem 1.15. *Let G be a finite group, $X \le M$ be normal subgroups of G with p is a prime divisor of $|M|$ and $F_p^*(M) \le X \le M$, $e \ge 3$ be an integer, and $P \in Syl_p(X)$ with $|P| \ge p^{e+1}$ and P is noncyclic. Suppose that for any noncyclic subgroup P_1 of P with order p^e, $P_1 \in \prod_{X_G^{*p}}(G)$. Then every chief factor of $G/O_{p'}(M)$ below $M/O_{p'}(M)$ is cyclic. In particular, every p-chief factor of G below M is cyclic.*

Theorem 1.16. *Let G be a finite group, $X \le M$ be normal subgroups of G with p is a prime divisor of $|M|$ and $F_p^*(M) \le X \le M$, and $P \in Syl_p(X)$ with $|P| \ge p^3$ and P is noncyclic. Suppose that for any subgroup P_1 of P with order p^2, $P_1 \in \prod_{X_G^{*p}}(G)$. Then every chief factor of $G/O_{p'}(M)$ below $M/O_{p'}(M)$ is cyclic. In particular, every p-chief factor of G below M is cyclic.*

Theorem 1.17. *Let G be a finite group, $X \le M$ be normal subgroups of G with p is a prime divisor of $|M|$ and $F_p^*(M) \le X \le M$, and $P \in Syl_p(X)$ with P is noncyclic. Suppose that for any subgroup $P_1 \in \mathbb{L}_1(P)$, $P_1 \in \prod_{X_G^{*p}}(G)$. Then every chief factor of $G/O_{p'}(M)$ below $M/O_{p'}(M)$ is cyclic. In particular, every p-chief factor of G below M is cyclic.*

We mention that Theorem 1.12–1.17 generalize the main results of [1], [3], [9], [10] and [12].

2. Preliminaries

Lemma 2.1 (1, Lemma 2.1(*b*)). *Let p be a prime and P be a nonidentity finite p-group. Let A act on P via automorphisms. Assume that P has a cyclic maximal subgroup, and P is neither elementary abelian of order p^2 nor isomorphic to Q_8. Then $O^p(A_p^*)$ acts trivially on P.*

Lemma 2.2 (1, Lemma 2.2). *Let S be a p-group for some odd prime p, $e \geq 2$ be an integer and $P \trianglelefteq S$ with $|P| \geq p^e$. Suppose that every normal subgroup of S that has order p^e and is contained in P is cyclic. Then P is cyclic.*

Lemma 2.3 (1, Lemma 2.3). *Fix an integer $e \geq 3$, and let S be a p-group with $|S| > p^e$. The following then hold.*
(1) *If every subgroup of order p^e in S is cyclic, then S is cyclic.*
(2) *If S has exactly one noncyclic subgroup P with order p^e, then P is abelian and has a cyclic maximal subgroup.*

By Problem 5C.12 of [5], we have the following lemma.

Lemma 2.4. *Let p be a prime dividing the order of a finite group G, $P \in Syl_p(G)$ and $N \trianglelefteq G$. Assume that P is cyclic and $P \cap N < P$. Then N is p-nilpotent.*

Lemma 2.5. *Let p be a prime dividing the order of a finite group G and $P \in Syl_p(G)$. Suppose that P is cyclic and there exists $1 < H \leq P$ such that H^G is p-solvable. Then G is p-supersolvable.*

Proof. It is no loss to assume that $O_{p'}(G) = 1$ and $P \nleq H^G$. By Lemma 2.4, it follow that H^G is p-nilpotent, and thus $H > 1$ is a normal p-subgroup of G. Hence $C_P(G_p^*) > 1$. Note that P is a cyclic p-subgroup, by Fitting's Theorem, it is not very difficult to see that G_p^* is p-nilpotent, i.e., G is p-supersolvable. □

Lemma 2.6. *Let p be a prime dividing the order of a finite group G, e be an integer, $N < M$ be normal subgroups of G, $S \in Syl_p(G)$, $P = S \cap M$, and $N = V \rtimes K$ with $V > 1$ is the normal Sylow p-subgroup of N and $K > 1$ is a Hall p'-subgroup of N. Assume that $|P| \geq p^{e+1}$ and $|V| \leq p^e$. Let $V_1 < V$ such that $V_1 \trianglelefteq G$ and V/V_1 is a chief factor of G. Suppose that for any normal noncyclic subgroup P_1 of S that has order p^e and is contained in P (if S has such a subgroup), $[G/V_1 : N_{G/V_1}((P_1 \cap V)V_1/V_1)]$ is a p-number. If N/V_1 is not p-nilpotent, then $|V/V_1| = p$.*

Proof. Consider $\overline{G} = G/V_1$. By Frattini's argument, It follows that $\overline{G} = N_{\overline{G}}(\overline{K})\overline{V}$. Hence $\overline{S} = N_{\overline{S}}(\overline{K})\overline{V}$. Since $\overline{N}$ is not p-nilpotent, we see that $N_{\overline{S}}(\overline{K}) < \overline{S}$. Hence S has a maximal subgroup T such that $V_1 \leq T$ and $N_{\overline{S}}(\overline{K}) \leq \overline{T}$. Hence $\overline{S} = \overline{TV}$ and $\overline{T} = N_{\overline{S}}(\overline{K})\overline{V \cap T}$. It is not very difficult to see that $[\overline{V} : \overline{V \cap T}] = [\overline{S} : \overline{T}] = p$. Let $|V_1| = p^f$. Then $f < e$. Note that $|\overline{V \cap T}| < |\overline{V}| \leq p^{e-f} \leq |\overline{P}|/p \leq |\overline{P \cap T}|$ and V, $P \cap T$ are normal subgroups of S. Hence there exists $V_1 < P_1 < S$ such that $P_1 \trianglelefteq S$, $|\overline{P_1}| = p^{e-f}$ and $\overline{V \cap T} < \overline{P_1} \leq \overline{P \cap T}$. Then $\overline{V \cap T} = \overline{V \cap P_1}$ and $|P_1| = p^e$.

If $\overline{P_1}$ is noncyclic, then P_1 is noncyclic, and thus P_1 is a normal noncyclic subgroup of S that has order p^e and is contained in P. Hence $[\overline{G} : N_{\overline{G}}(\overline{V \cap P_1})]$ is a p-number. Hence $\overline{G} = N_{\overline{G}}(\overline{V \cap P_1})\overline{S}$. Note that $\overline{V \cap T} = \overline{V \cap P_1} \trianglelefteq \overline{S}$. Then $\overline{V \cap T} = \overline{V \cap P_1} \trianglelefteq \overline{G}$.

Assume that $\overline{P_1}$ is cyclic. Since $\overline{T} = N_{\overline{S}}(\overline{K})\overline{V \cap T}$ and $\overline{V \cap T} < \overline{P_1}$, it follows that $\overline{P_1} = N_{\overline{P_1}}(\overline{K})\overline{V \cap T}$. Hence $\overline{P_1} = N_{\overline{P_1}}(\overline{K})$. Hence $\overline{V \cap T} = \overline{V \cap P_1} \leq N_{\overline{V}}(\overline{K}) < \overline{V}$. Since $[\overline{V} : \overline{V \cap T}] = p$, it follows that $\overline{V \cap T} = N_{\overline{V}}(\overline{K})$. Hence $\overline{V \cap T} \trianglelefteq N_{\overline{G}}(\overline{K})$. Note that $\overline{V \cap T} \trianglelefteq \overline{V}$. Hence $\overline{V \cap T} \trianglelefteq N_{\overline{G}}(\overline{K})\overline{V} = \overline{G}$.

Since $[\overline{V} : \overline{V \cap T}] = p$ and $\overline{V}$ is a minimal normal subgroup of $\overline{G}$, it follows that $\overline{V \cap T} = 1$. Hence $|\overline{V}| = p$. □

Lemma 2.7. *Let p be a prime and P be a nonidentity finite p-group. Let $1 < N \leq P$ be such that $N \cap \Phi(P) = 1$. Then for any maximal subgroup N_1 of N, there exists a maximal subgroup T of P such that $N_1 = T \cap N$.*

Proof. Consider $\overline{P} = P/\Phi(P)$. Since $\overline{P}$ is an elementary abelian p-group, there exists $\Phi(P) \leq M \leq P$ such that $\overline{P} = \overline{N} \times \overline{M}$. Hence $M \trianglelefteq P$, $P = (N\Phi(P))M = NM$ and $(N\Phi(P)) \cap M = \Phi(P)$. Hence $N \cap M \leq (N\Phi(P)) \cap M = \Phi(P)$, and thus $N \cap M = N \cap \Phi(P) = 1$. Since $N > 1$ and $N \cap M = 1$, it follows that $P/M = NM/M \cong N > 1$. Recall that N_1 is a maximal subgroup of N, it is not very difficult to see that N_1M is a maximal subgroup of P. Let $T = N_1M$. Then $N \cap T = N_1(N \cap M) = N_1$. □

Lemma 2.8 (1, Lemma 3.6). *Suppose that a finite group G acts irreducibly on an elementary abelian p-group V, and assume that $O^p(G_p^*)$ acts trivially on V. Then $|V| = p$.*

Lemma 2.9. *Let p be a prime dividing the order of a finite group G and H be an S-semipermutable p-subgroup of G. Then H satisfies Π-property in G.*

Proof. Let K/L be a chief factor of G. Consider $\overline{G} = G/L$. We work to prove that $O^p(\overline{G})$ normalizes $\overline{H \cap K}$. It is no loss to assume that $\overline{H \cap K} > 1$. Since H is an S-semipermutable p-subgroup of G, it is not very difficult to see that $\overline{H \cap K} = \overline{H} \cap \overline{K}$ is S-semipermutable in $\overline{G}$. By Theorem A of [6], it follows that $(\overline{H \cap K})^{\overline{G}}$ is solvable. Recall that $1 < \overline{H \cap K} \leq \overline{K}$ and $\overline{K}$ is a minimal normal subgroup of $\overline{G}$. Hence $\overline{K} = (\overline{H \cap K})^{\overline{G}}$ is solvable. Then $\overline{K}$ is a p-subgroup. By Lemma 3.2 of [1], it follows that $O^p(\overline{G})$ normalizes $\overline{H \cap K}$. In particular, $[\overline{G} : N_{\overline{G}}(\overline{H \cap K})]$ is a p-number. By the definition of Π-property of subgroups of finite groups, we see that H satisfies Π-property in G. □

Lemma 2.10 (8, Theorem C). *Let G be a finite group and $1 < M \trianglelefteq G$. Suppose that every chief factor of G below $F^*(M)$ is cyclic. Then every chief factor of G below M is cyclic.*

Lemma 2.11. *Let p be a prime dividing the order of a finite group G and $1 < M \trianglelefteq G$. Suppose that $F^*(M)$ is p-solvable and $O_{p'}(M) = 1$. If every p-chief factor of G below $F^*(M)$ is cyclic, then every chief factor of G below M is cyclic.*

Proof. Assume that there exists $H \trianglelefteq\trianglelefteq M$ such that $H/Z(H)$ is a nonabelian simple group and $H' = H$. Since $H \leq F^*(M)$ and $F^*(M)$ is p-solvable, it follows that $H/Z(H)$ is p-solvable. Recall that $H/Z(H)$ is a nonabelian simple group. Hence $H/Z(H)$ is a p'-group. Let $P_1 \in Syl_p(H)$. Since $H/Z(H)$ is a p'-group, it follows that $P_1 \leq Z(H)$. By Burnside's Theorem (see Theorem 5.13 of [5]), it follows that H is p-nilpotent. Since $H \trianglelefteq\trianglelefteq M$ and $O_{p'}(M) = 1$, we have $O_{p'}(H) = 1$. Hence $H = P_1$ is a p-group. This is a contradiction since $H/Z(H)$ is a nonabelian simple group. Hence $F^*(M) = F(M)$. Recall that $O_{p'}(M) = 1$. Then $F^*(M) = O_p(M)$.

Since every p-chief factor of G below $F^*(M) = O_p(M)$ is cyclic, it follows that every chief factor of G below $F^*(M)$ is cyclic. By Lemma 2.10, every chief factor of G below M is cyclic. □

3. Main Results

Theorem 3.1. *Let G be a finite group and $M \trianglelefteq G$. Suppose that M_G^{*2} has a cyclic Sylow 2-subgroup. Then every 2-chief factor of G below M is cyclic.*

Proof. Since M_G^{*2} has a cyclic Sylow 2-subgroup, by Corollary 5.14 of [5], it follows that M_G^{*2} is 2-nilpotent. Hence every 2-chief factor of G below M_G^{*2} is cyclic, and thus every 2-chief factor of G below M is cyclic. □

Theorem 3.2. *Let G be a finite group, $M \trianglelefteq G$ with $p > 2$ is a prime divisor of $|M|$, $S \in Syl_p(G)$ and $e \geq 2$ be an integer. Let $P = S \cap M$. Assume that $|P| \geq p^e$, P is noncyclic and $P \cap M_G^{*p}$ is cyclic. Suppose that for any normal noncyclic subgroup P_1 of S that has order p^e and is contained in P (by Lemma 2.2, we see that S has such a subgroup), $P_1 \in \prod_{M_G^{*p}}(G)$. Then every p-chief factor of G below M is cyclic.*

Proof. Suppose that M is a counterexample with minimal order and we work to obtain a contradiction. Then $M_G^{*p} > 1$.

It is no loss to assume that $O_{p'}(M) = 1$. To see this, assume that $O_{p'}(M) > 1$ and we work to obtain a contradiction. Consider $G/O_{p'}(M)$. It is not very difficult to see that the hypotheses are inherited by $M/O_{p'}(M)$. By induction, we see that every p-chief factor of $G/O_{p'}(M)$ below $M/O_{p'}(M)$ is cyclic, and thus every p-chief factor of G below M is cyclic. This is a contradiction.

Let $N > 1$ be a minimal normal subgroup of G that is contained in M_G^{*p}. Since $O_{p'}(M) = 1$, it follows that $P \cap N > 1$. We claim that S has a normal noncyclic subgroup P_1 that has order p^e and is contained in P such that $(P \cap N) \cap P_1 > 1$. By Lemma 2.2, we see that S has a normal noncyclic subgroup N_1 that has order p^e and is contained in P. Assume that $(P \cap N) \cap N_1 > 1$. Let $P_1 = N_1$. Then P_1 is a normal noncyclic subgroup of S that has order p^e and is contained in P such that $(P \cap N) \cap P_1 > 1$. Assume that $(P \cap N) \cap N_1 = 1$. Let Z_1 be the subgroup of $P \cap N$ with order p. Since $P \cap N$ is cyclic, we see that $Z_1 \trianglelefteq S$. Since $N_1 \trianglelefteq S$ and $N_1 > 1$, N_1 has a maximal subgroup Z_2 such that $Z_2 \trianglelefteq S$. Then $|Z_2| = p^{e-1} \geq p$. From

$(P \cap N) \cap N_1 = 1$, we see that $Z_1 \cap Z_2 = 1$. Let $P_1 = Z_1 \times Z_2$. Then P_1 is a normal noncyclic subgroup of S that has order p^e and is contained in P such that $(P \cap N) \cap P_1 = Z_1 > 1$.

Let P_1 be a normal noncyclic subgroup of S that has order p^e and is contained in P such that $(P \cap N) \cap P_1 > 1$. Note that N is a minimal normal subgroup of G. Since $P_1 \in \prod_{M_G^{*p}}(G)$, we see that $[G : N_G(P_1 \cap N)]$ is a p-number. Hence $G = N_G(P_1 \cap N)S$. Note that $P_1 \cap N \trianglelefteq S$. Hence $1 < P_1 \cap N \trianglelefteq G$. By Lemma 2.5, it follows that M_G^{*p} is p-supersolvable. Hence every p-chief factor of G below M_G^{*p} is cyclic, and thus every p-chief factor of G below M is cyclic. This is a contradiction. □

Theorem 3.3. *Let G be a finite group, $M \trianglelefteq G$ with $p > 2$ is a prime divisor of $|M|$ and $P \in Syl_p(M)$. Assume that P is cyclic and there exists $1 < P_1 \leq P$ such that $P_1 \in \prod_{M_G^{*p}}(G)$. Then every p-chief factor of G below M is cyclic.*

Proof. Suppose that M is a counterexample with minimal order and we work to obtain a contradiction. Then $M_G^{*p} > 1$. Let $S \in Syl_p(G)$ such that $P \leq S$.

It is no loss to assume that $O_{p'}(M) = 1$. To see this, assume that $O_{p'}(M) > 1$ and we work to obtain a contradiction. Consider $G/O_{p'}(M)$. It is not very difficult to see that the hypotheses are inherited by $M/O_{p'}(M)$. By induction, we see that every p-chief factor of $G/O_{p'}(M)$ below $M/O_{p'}(M)$ is cyclic, and thus every p-chief factor of G below M is cyclic. This is a contradiction.

Let $N > 1$ be a minimal normal subgroup of G that is contained in M_G^{*p}. Since $O_{p'}(M) = 1$, it follows that $P \cap N > 1$. Note that P is a cyclic p-subgroup and $P \cap N$, P_1 are nontrivial subgroups of P. Hence $P_1 \cap N = P_1 \cap (P \cap N) > 1$. Since $1 < N \leq M_G^{*p}$ and N is a minimal normal subgroup of G, by $P_1 \in \prod_{M_G^{*p}}(G)$, it follows that $[G : N_G(P_1 \cap N)]$ is a p-number. Hence $G = N_G(P_1 \cap N)S$. Note that $P_1 \cap N \trianglelefteq S$. Hence $P_1 \cap N \trianglelefteq G$. By Lemma 2.5, it follows that M_G^{*p} is p-supersolvable. Hence every p-chief factor of G below M_G^{*p} is cyclic, and thus every p-chief factor of G below M is cyclic. This is a contradiction. □

Theorem 3.4. *Let p be a prime dividing the order of a finite group G and $1 < P \trianglelefteq G$ is a p-subgroup. Suppose that for any maximal subgroup P_1 of P, $P_1 \in \prod_{P_G^{*p}}(G)$. Then every chief factor of G below P is cyclic.*

Proof. Suppose that P is a counterexample with minimal order and we work to obtain a contradiction. Then $P_G^{*p} > 1$. Let $N > 1$ be a minimal normal subgroup of G that is contained in P_G^{*p}. We claim that $N = P_G^{*p}$. Assume that $N < P_G^{*p}$ and we work to obtain a contradiction. Consider G/N. It is not very difficult to see that the hypotheses are inherited by P/N. By induction, it follows that every chief factor of G/N below P/N is cyclic, and thus $P_G^{*p} \leq N$. This is a contradiction. Hence $P_G^{*p} = N$ is a minimal normal subgroup of G.

We claim that $P_G^{*p} \cap \Phi(P) = 1$. Assume that $P_G^{*p} \cap \Phi(P) > 1$ and we work to obtain a contradiction. Since P_G^{*p} is a minimal normal subgroup of G, we see that $P_G^{*p} \leq \Phi(P)$. Note that every chief factor of G/P_G^{*p} below P/P_G^{*p} is cyclic, by Corollary 3.28 of [5], we see that P/P_G^{*p} is centralized by $O^p(G_p^*)$. By Corollary 3.29 of [5], we see that P is centralized by $O^p(G_p^*)$. By Lemma 2.8, it follows that every chief factor of G below P is cyclic. This is a contradiction. Hence $P_G^{*p} \cap \Phi(P) = 1$. Let $S \in Syl_p(G)$. Then $P \leq S$. Since $1 < P_G^{*p} \trianglelefteq S$, P_G^{*p} has a maximal subgroup N_1 such that $N_1 \trianglelefteq S$. By Lemma 2.7, it follows that P has a maximal subgroup P_1 such that $N_1 = P_1 \cap P_G^{*p}$. Since P_G^{*p} is a minimal normal subgroup of G and $P_1 \in \prod_{P_G^{*p}}(G)$, it follows that $[G : N_G(N_1)] = [G : N_G(P_1 \cap P_G^{*p})]$ is a p-number. Hence $G = N_G(N_1)S$. Recall that $N_1 \trianglelefteq S$. Hence $N_1 \trianglelefteq G$. Since P_G^{*p} is a minimal normal subgroup of G and $[P_G^{*p} : N_1] = p$, we see that $N_1 = 1$ and $|P_G^{*p}| = p$. Since every chief factor of G/P_G^{*p} below P/P_G^{*p} is cyclic, it follows that every chief factor of G below P is cyclic. This is a contradiction. □

THEOREM 3.5. *Let p be a prime dividing the order of a finite group G, $e \geq 3$ be an integer, and $1 < P \trianglelefteq G$ be a p-subgroup with $|P| \geq p^{e+1}$ and P is noncyclic. Suppose that for any noncyclic subgroup P_1 of P with order p^e (by Lemma 2.3(1), P has such a subgroup), $P_1 \in \prod_P(G)$. Then every chief factor of G below P is cyclic.*

PROOF. Suppose that P is a counterexample with minimal order and we work in the following steps to obtain a contradiction. Let $B = O^p(G_p^*)$ and $C = C_P(B)$. By Lemma 2.8, it follows that $C < P$. Let $S \in Syl_p(G)$. Then $P \leq S$. Let $\Omega = \{H < P, H \trianglelefteq G \mid P/H$ is a chief factor of $G\}$. Since $1 < P \trianglelefteq G$, it is not very difficult to see that Ω is not empty.

Step 1 $|P| > p^{e+1}$.

Assume that $|P| \leq p^{e+1}$ and we work to obtain a contradiction. Recall that $|P| \geq p^{e+1}$. Hence $|P| = p^{e+1}$, and thus for any maximal subgroup P_1 of P, $|P_1| = p^e$. If every maximal subgroup of P is noncyclic, by Theorem 3.4, it follows that every chief factor of G below P is cyclic. This is a contradiction. Hence P has a cyclic maximal subgroup. Note that $|P| = p^{e+1} \geq p^4$, by Lemma 2.1, it follows that P is centralized by B, i.e., $P \leq C$. This is a contradiction.

Step 2 For any $H \in \Omega$, we have $H \leq C$.

If H is cyclic, it is not very difficult to see that $H \leq C$.

If H is noncyclic and $|H| \geq p^{e+1}$, it is not very difficult to see that the hypotheses are inherited by H. By induction, it follows that $H \leq C$.

Assume that H is noncyclic and $|H| \leq p^e$. Since H, P are normal subgroups of S and $|H| \leq p^e < p^{e+1} \leq |P|$, we see that S has a normal subgroup P_1 with order p^e and a normal subgroup P_2 with order p^{e+1} such that $H \leq P_1 < P_2 \leq P$. Since H is noncyclic, we see that P_1 is noncyclic. Since $P_1 \in \prod_P(G)$ and P/H is a chief factor of G, it follows that $[G/H : N_{G/H}(P_1/H)]$ is a p-number. Hence $G/H = N_{G/H}(P_1/H)S/H$. Recall that $P_1 \trianglelefteq S$. Hence $P_1/H \trianglelefteq G/H$, and thus $P_1 \trianglelefteq G$. Note that $H \leq P_1 < P$ and P/H is a chief factor of G. Hence $H = P_1$, and thus $|H| = p^e$. Hence $H = P_1$ is a noncyclic maximal subgroup of P_2. We claim that H is the unique noncyclic maximal subgroup of P_2. Assume that P_2 has another noncyclic maximal subgroup P_3 and we work to obtain a contradiction. Then

$P_2 = P_3H$. Since $P_3 \in \prod_P(G)$ and P/H is a chief factor of G, it follows that $[G/H : N_{G/H}(P_2/H)] = [G/H : N_{G/H}(P_3H/H)]$ is a p-number. Hence $G/H = N_{G/H}(P_2/H)S/H$. Recall that $P_2 \trianglelefteq S$. Hence $P_2/H \trianglelefteq G/H$, and thus $P_2 \trianglelefteq G$. By Step 1, we see that $H < P_2 < P$. Recall that P/H is a chief factor G. Hence we obtain a contradiction. Hence H is the unique noncyclic maximal subgroup of P_2. Note that $e \geq 3$ and $|H| = p^e < p^{e+1} = |P_2|$, by Lemma 2.3(2), it follows that H is abelian and H has a cyclic maximal subgroup. Note that $|H| = p^e \geq p^3$. By Lemma 2.1, we see that $H \leq C$.

Step 3 $\Omega = \{C\}$. And if $N < P$ such that $N \trianglelefteq G$, then $N \leq C$.

For any $H \in \Omega$, by Step 2, it follows that $H \leq C$. Since $H \leq C < P$, $C \trianglelefteq G$ and P/H is a chief factor of G, we see that $C = H$. Hence $\Omega = \{C\}$.

If $N < P$ such that $N \trianglelefteq G$, then there exists $T \in \Omega$ such that $N \leq T$. Since $\Omega = \{C\}$, we see that $N \leq C$.

Step 4 $P = \{x \in P \mid x^{p^2} = 1\}$. Hence every subgroup of P with order p^e is noncyclic.

Note that $\Phi(P) < P$ and $\Phi(P) \trianglelefteq G$, by Step 3, we see that $\Phi(P) \leq C$. Note that $[P, B] \leq P$ and $[P, B] \trianglelefteq G$. If $[P, B] < P$, by Step 3, we see that $[P, B] \leq C$, i.e., $[P, B, B] = 1$. By Lemma 4.29 of [5], we see that $[P, B] = 1$, i.e., $P \leq C$. This is a contradiction. Hence $[P, B] = P$. Since $[\Phi(P), B, P] = 1$ and $[P, \Phi(P), B] = 1$, by Hall's three-subgroups Lemma (see Lemma 4.9 of [5]), we see that $[P, \Phi(P)] = [B, P, \Phi(P)] = 1$, i.e., $\Phi(P) \leq Z(P)$. Let $U = \{x \in P \mid x^{p^2} = 1\}$. Since $\Phi(P) \leq Z(P)$, it is not very difficult to prove that U is a subgroup of P. To see this, for any $x, y \in U$, by $P' \leq \Phi(P) \leq Z(P)$, we see that $(xy)^{p^2} = x^{p^2}y^{p^2}[y, x]^{p^2(p^2-1)/2} = [y^{p^2(p^2-1)/2}, x]$. Since p divides $p^2(p^2-1)/2$, we see that $y^{p^2(p^2-1)/2} \in \Phi(P) \leq Z(P)$. Hence $(xy)^{p^2} = [y^{p^2(p^2-1)/2}, x] = 1$, and thus $xy \in U$. Hence $U \leq P$. Furthermore, we have $U \trianglelefteq G$. If $U < P$, by Step 3, we see that $U \leq C$. By Satz IV.5.12 of [4], it follows that P is centralized by B, i.e., $P \leq C$. This is a contradiction. Hence $P = U$. Note that $e \geq 3$. Hence every subgroup of P with order p^e is noncyclic.

Step 5 $|C| \geq p^e$.

Assume that $|C| < p^e$ and we work to obtain a contradiction. Since C, $P \trianglelefteq S$ and $|C| < p^e < |P|$, S has a normal subgroup P_4 with order p^e such that $C < P_4 < P$. By Step 4, it follows that P_4 is noncyclic, and thus $P_4 \in \prod_P(G)$. By Step 3, we see that $[G/C : N_{G/C}(P_4/C)]$ is a p-number. Hence $G/C = N_{G/C}(P_4/C)S/C$. Recall that $P_4 \trianglelefteq S$. Hence $P_4/C \trianglelefteq G/C$, and thus $P_4 \trianglelefteq G$. Note that $C < P_4 < P$ and P/C is a chief factor of G. This is a contradiction. Hence $|C| \geq p^e$.

Step 6 The final contradiction.

Since C, $P \trianglelefteq S$ and $C < P$, S has a normal subgroup C_1 such that $C < C_1 \leq P$ and $|C_1/C| = p$. For any $x \in C_1 \backslash C$, by $|C_1/C| = p$, it follows that $C_1 = \langle x \rangle C$. By Step 4, we see that $|\langle x \rangle| \leq p^2$. By Step 5, it follows that $|\langle x \rangle| \leq p^2 < p^e \leq |C| < |\langle x \rangle C| = |C_1|$. Hence P has a subgroup P_5 with order p^e such that $\langle x \rangle < P_5 < C_1$. Hence $C_1 = P_5C$. By Step 4, we see that P_5 is noncyclic, and thus $P_5 \in \prod_P(G)$. Hence $[G/C : N_{G/C}(C_1/C)] = [G/C : N_{G/C}(P_5C/C)]$ is a p-number. Hence $G/C = N_{G/C}(C_1/C)S/C$. Recall that $C_1 \trianglelefteq S$. Hence $C_1/C \trianglelefteq G/C$, and thus $C_1 \trianglelefteq G$. Note that $C < C_1 \leq P$ and P/C is a chief factor of G. Then $P = C_1$, and thus $|P/C| = p$. Hence P/C is centralized by B. By Corollary 3.28 of [5], it follows that P is centralized by B, i.e., $P \leq C$. This is the final contradiction. □

Mimic the proof of Theorem 3.5, we can prove the following two results.

THEOREM 3.6. *Let p be a prime dividing the order of a finite group G and $1 < P \trianglelefteq G$ be a p-subgroup with $|P| \geq p^3$ and P is noncyclic. Suppose that for any subgroup P_1 of P with order p^2, $P_1 \in \prod_P(G)$. Then every chief factor of G below P is cyclic.*

THEOREM 3.7. *Let p be a prime dividing the order of a finite group G and $1 < P \trianglelefteq G$ be a p-subgroup with P is noncyclic. Suppose that for any $P_1 \in \mathbb{L}_1(P)$, $P_1 \in \prod_P(G)$. Then every chief factor of G below P is cyclic.*

THEOREM 3.8. *Let G be a finite group, $M \trianglelefteq G$ with p is a prime divisor of $|M|$, $e \geq 3$ be an integer, and $P \in Syl_p(M)$ with $|P| \geq p^{e+1}$ and P is noncyclic. Suppose that for any noncyclic subgroup P_1 of P with order p^e (by Lemma 2.3(1), P has such a subgroup), $P_1 \in \prod_M(G)$. Then every p-chief factor of G below M is cyclic.*

PROOF. Suppose that M is a counterexample with minimal order and we work in the following steps to obtain a contradiction. Then $M_G^{*p} > 1$. Let $S \in Syl_p(G)$ such that $P \leq S$. Let $\Omega = \{H < M,\ H \trianglelefteq G \mid M/H$ is a chief factor of $G\}$. Since $1 < M \trianglelefteq G$, we see that Ω is not empty.

Step 1 $O_{p'}(M) = 1$ and $O^{p'}(M) = M$.

Assume that $O_{p'}(M) > 1$ and we work to obtain a contradiction. Consider $G/O_{p'}(M)$. It is not very difficult to see that the hypotheses are inherited by $M/O_{p'}(M)$. By induction, we see that every p-chief factor of $G/O_{p'}(M)$ below $M/O_{p'}(M)$ is cyclic, and thus every p-chief factor of G below M is cyclic. This is a contradiction.

Assume that $O^{p'}(M) < M$ and we work to obtain a contradiction. It is not very difficult to see that the hypotheses are inherited by $O^{p'}(M)$. By induction, we see that every p-chief factor of G below $O^{p'}(M)$ is cyclic, and thus every p-chief factor of G below M is cyclic. This is a contradiction.

Step 2 For any $H \in \Omega$, H is p-solvable.

If $P \cap H$ is noncyclic and $|P \cap H| \geq p^{e+1}$, it is not very difficult to see that the hypotheses are inherited by H. By induction, we see that every p-chief factor of G below H is cyclic. In particular, H is p-solvable.

If $P \cap H$ is noncyclic and $|P \cap H| \leq p^e$. Note that $|P \cap H| \leq p^e < |P|$. Then P has a subgroup P_1 with order p^e such that $P \cap H \leq P_1 < P$. Since $P \cap H$ is noncyclic, it follows that P_1 is noncyclic, and thus $P_1 \in \prod_M(G)$. For any chief factor K/L of G below H, $(P_1 \cap K)L/L = (P \cap K)L/L \in Syl_p(K/L)$. Hence $[G/L : N_{G/L}((P \cap K)L/L)]$ is a p-number. Hence $[K/L : N_{K/L}((P \cap K)L/L)]$ is a p-number, and thus $(P \cap K)L/L \trianglelefteq K/L$. Hence K/L is p-solvable. Then H is p-solvable.

Assume that $P \cap H$ is cyclic. It is no loss to assume that $H > 1$. Let $N > 1$ be a minimal normal subgroup of G that is contained in H. By Step 1, we have $P \cap N > 1$. We claim that P has a noncyclic subgroup P_1 with order p^e such that $(P \cap N) \cap P_1 > 1$. Note that $e \geq 3$ and $|P| \geq p^{e+1} > p^e$. By Lemma 2.3(1), P has a noncyclic subgroup N_1 with order p^e. Assume that $(P \cap N) \cap N_1 > 1$. Let $P_1 = N_1$. Then P_1 is a noncyclic subgroup of P with order p^e such that $(P \cap N) \cap P_1 > 1$. Assume that $(P \cap N) \cap N_1 = 1$. Let Z_1 be the subgroup of $P \cap N > 1$ with order p. Since $P \cap N$ is cyclic, we see that $Z_1 \trianglelefteq P$, and thus $Z_1 \leq Z(P)$. Note that $N_1 > 1$. Let Z_2 be a maximal subgroup of N_1. Then $|Z_2| = p^{e-1} \geq p^2$. Note that $[Z_1, Z_2] = 1$. From $(P \cap N) \cap N_1 = 1$, we see that $Z_1 \cap Z_2 = 1$. Let $P_1 = Z_1 \times Z_2$. Then P_1 is a noncyclic subgroup of P with order p^e and $(P \cap N) \cap P_1 = Z_1 > 1$. Let P_1 be a noncyclic subgroup of P with order p^e such that $(P \cap N) \cap P_1 > 1$. Note that $N < M$ and N is a minimal normal subgroup of G. Then $[G : N_G(P_1 \cap N)]$ is a p-number. Hence $G = N_G(P_1 \cap N)S$, and thus $1 < (P_1 \cap N)^G \leq S$ is p-subgroup. By Lemma 2.5, we see that H is p-supersolvable.

Step 3 For any noncyclic subgroup P_1 of P with order p^e, P_1^G is p-solvable.

Let $H \in \Omega$. We consider $\overline{G} = G/H$. Since $P_1 \in \prod_M(G)$ and M/H is a chief factor of G, we have $[\overline{G} : N_{\overline{G}}(\overline{P_1})]$ is p-number. Then $\overline{G} = N_{\overline{G}}(\overline{P_1})\overline{S}$. Hence $\overline{P_1^G} = (\overline{P_1})^{\overline{G}} \leq \overline{S}$ is a p-subgroup. By Step 2, it follows that P_1^G is p-solvable.

Step 4 Let $\Delta = \{P_1 \leq P \mid P_1$ is a noncyclic subgroup with order $p^e\}$ (by Lemma 2.3(1), Δ is not empty). Let $W = \prod_{P_1 \in \Delta} P_1^G$. Then W is not a p-subgroup and $|O_p(W)| \leq p^e$.

By Step 3, we see that W is p-solvable and $|W| \geq p^e$. Note that $W \leq M$ and $W \trianglelefteq G$. By Step 1, it follows that $O_{p'}(W) = 1$. Recall that $W > 1$ and W is p-solvable. Hence $O_p(W) > 1$.

Assume that W is a p-subgroup and we work to obtain a contradiction. We claim that W is centralized by $O^p(M)$. If W is a cyclic p-subgroup, it is not very difficult to see that W is centralized by $O^p(G_p^*)$. By Step 1, we have $M_p^* = M$, and thus W is centralized by $O^p(M)$. If W is a noncyclic p-subgoup and $|W| \geq p^{e+1}$, by Theorem 3.5, W is centralized by $O^p(G_p^*)$, and thus W is centralized by $O^p(M)$. If W is a noncyclic p-subgroup and $|W| \leq p^e$, since $|W| \geq p^e$, it follows that $|W| = p^e$. Hence W is the unique noncyclic subgroup of P with order p^e. Recall that $e \geq 3$ and $|P| \geq p^{e+1}$, by Lemma 2.3(2), we see that W is abelian and W has a cyclic maximal subgroup. Recall that $|W| = p^e > p^2$. We see that W is neither elementary abelian of order p^2 nor isomorphic to Q_8, and thus W is centralized by $O^p(G_p^*)$. Then W is centralized by $O^p(M)$. Now we claim that for any subgroup X of P with $|X| < p^e$, we have $X \leq W$. Let $X \leq P$ with $|X| < p^e$. Then $|X| < p^e \leq |W| \leq |WX|$. Hence there exists $Y \leq P$ such that $|Y| = p^e$ and $X < Y \leq WX$. Then $Y = (Y \cap W)X$. If Y is cyclic, since $X < Y$, we see that $Y = Y \cap W \leq W$, and thus $X < Y \leq W$. If Y is noncyclic, then $X < Y \leq W$. Recall that $e \geq 3$. Then for any $x \in P$ such that the order of x divides p^2, we have $\langle x \rangle \leq W$. Hence $\langle x \rangle$ is centralized by $O^p(M)$. By Frobenius' Theorem (see Theorem 5.26 of [5]) and Satz IV.5.12 of [4], it follows that M is

p-nilpotent. By Step 1, we have $M = P$. By Theorem 3.5, it follows that every p-chief factor of G below $M = P$ is cyclic. This is a contradiction.

Assume that $|O_p(W)| \geq p^{e+1}$ and we work to obtain a contradiction. If $O_p(W)$ is cyclic, we see that $O_p(W)$ is centralized by $O^p(G_p^*)$. If $O_p(W)$ is noncyclic, by Theorem 3.5, we see that $O_p(W)$ is centralized by $O^p(G_p^*)$. Hence $O_p(W)$ is centralized by $O^p(M)$, and thus $O_p(W)$ is centralized by $O^p(W)$. Since W is p-solvable and $O_{p'}(W) = 1$, by Hall-Higman's Lemma (see Theorem 3.21 of [5]), we see that $O^p(W) \leq C_W(O_p(W)) \leq O_p(W)$. Hence $O^p(W) = 1$, i.e., W is a p-subgroup. This is a contradiction.

Step 5 Let $O_{p,p'}(W)$ be the subgroup such that $O_p(W) \leq O_{p,p'}(W)$ and $O_{p,p'}(W)/O_p(W) = O_{p'}(W/O_p(W))$. Let $R = O^p(O_{p,p'}(W))$. Then $R = V \rtimes K$ with $V > 1$ is the normal Sylow p-subgroup of R, $|V| \leq p^e$ and $K > 1$ is a Hall p'-subgroup of R.

By Step 4, we see that $O_p(W) < W$. Recall that W is p-solvable and $O_p(W) < W$, we see that $O_p(W) < O_{p,p'}(W)$. Let $K > 1$ be a Hall p'-subgroup of $O_{p,p'}(W)$. Then $O_{p,p'}(W) = O_p(W) \rtimes K$. Let $V = O_p(W) \cap R$. Then V is the normal Sylow p-subgroup of R and $R = V \rtimes K$. By Step 4, we see that $|V| \leq |O_p(W)| \leq p^e$. Since $O_{p'}(M) = 1$ (Step 1) and $O_{p,p'}(W)$ is not a p-subgroup, it follows that $O_{p,p'}(W)$ is not p-nilpotent, i.e., R is not a p'-subgroup. Hence $V > 1$.

Step 6 The final contradiction.

Let $V_1 < V$ be a normal subgroup of G such that V/V_1 is a chief factor of G. Since $R = O^p(O_{p,p'}(W))$, we have $O^p(R) = R$, and thus R/V_1 is not p-nilpotent. For any noncyclic subgroup P_1 of P with order p^e, we have $P_1 \in \prod_M(G)$. Note that V/V_1 is a chief factor of G below M. Then $[G/V_1 : N_{G/V_1}((P_1 \cap V)V_1/V_1)]$ is a p-number. By Lemma 2.6, we see that $|V/V_1| = p$. Hence V/V_1 is centralized by G_p^*. By Step 1, we see that $M_p^* = M$. Hence V/V_1 is centralized by M, and thus V/V_1 is centralized by R. Hence $V/V_1 \leq Z(R/V_1)$. By Burnside's Theorem (see Theorem 5.13 of [5]), it follows that R/V_1 is p-nilpotent. Recall that R/V_1 is not p-nilpotent. This is the final contradiction. □

Mimic the proof of Theorem 3.8, we can prove the following two results.

Theorem 3.9. *Let G be a finite group, $M \trianglelefteq G$ with p is a prime divisor of $|M|$, and $P \in Syl_p(M)$ with $|P| \geq p^3$ and P is noncyclic. Suppose that for any subgroup P_1 of P with order p^2, $P_1 \in \prod_M(G)$. Then every p-chief factor of G below M is cyclic.*

Theorem 3.10. *Let G be a finite group, $M \trianglelefteq G$ with p is a prime divisor of $|M|$, and $P \in Syl_p(M)$ with P is noncyclic. Suppose that for any $P_1 \in \mathbb{L}_1(P)$, $P_1 \in \prod_M(G)$. Then every p-chief factor of G below M is cyclic.*

Now we work to prove Theorem 1.12.

Proof of Theorem 1.12. Suppose that M is a counterexample with minimal order and we work in the following steps to obtain a contradiction. Then $M_G^{*p} > 1$.

Step 1 $O_{p'}(M) = 1$.

Assume that $O_{p'}(M) > 1$ and we work to obtain a contradiction. Consider $G/O_{p'}(M)$. It is not very difficult to see that the hypotheses are inherited by $M/O_{p'}(M)$. By induction, we see that every p-chief factor of $G/O_{p'}(M)$ below $M/O_{p'}(M)$ is cyclic, and thus every p-chief factor of G below M is cyclic. This is a contradiction.

Step 2 $P \cap M_G^{*p}$ is noncyclic.

Assume that $P \cap M_G^{*p}$ is cyclic, by Theorem 3.1 and Theorem 3.2, we see that every p-chief factor of G below M is cyclic. This is a contradiction.

Step 3 M_G^{*p} is a minimal normal subgroup of G and M_G^{*p} is an elementary abelian p-group.

At first, we work to prove that M_G^{*p} is p-solvable. Since $|P \cap M_G^{*p}| \leq p^e < |P|$, P has a normal subgroup P_1 with order p^e such that $P \cap M_G^{*p} \leq P_1 < P$. Then $P_1 \cap M_G^{*p} = P \cap M_G^{*p}$. By Step 2, we see that P_1 is noncyclic. Then $P_1 \in \prod_{M_G^{*p}}(G)$. For any chief factor K/L of G below M_G^{*p}, we have $(P_1 \cap K)L/L = (P \cap K)L/L \in Syl_p(K/L)$. Hence $[G/L : N_{G/L}((P \cap K)L/L)]$ is a p-number. Then $[K/L : N_{K/L}((P \cap K)L/L)]$ is a p-number, and thus $(P \cap K)L/L \trianglelefteq K/L$. Hence K/L is p-solvable. Then M_G^{*p} is p-solvable.

Let $N > 1$ be a minimal normal subgroup of G that is contained in M_G^{*p}. Since $M_G^{*p} > 1$ is p-solvable and $O_{p'}(M) = 1$, we see that N is an elementary abelian p-subgroup. Let $|N| = p^f$. Then $1 \leq f \leq e$. Consider $\overline{G} = G/N$. Then $|\overline{P}| \geq p^{e-f+1}$ and $|\overline{P} \cap \overline{M}_{\overline{G}}^{*p}| = |\overline{P \cap M_G^{*p}}| \leq p^{e-f}$. If $\overline{P \cap M_G^{*p}}$ is cyclic, since M_G^{*p} is p-solvable, it follows that $\overline{M}_{\overline{G}}^{*p}$ is p-supersolvable. Then every p-chief factor of $\overline{G}$ below $\overline{M}_{\overline{G}}^{*p}$ is cyclic. Hence every p-chief factor of $\overline{G}$ below $\overline{M}$ is cyclic, and thus $M_G^{*p} \leq N$. If $\overline{P \cap M_G^{*p}}$ is noncyclic, then $e - f \geq 2$. For any normal noncyclic subgroup $\overline{P_2}$ ($N < P_2$) of $\overline{P}$ with order p^{e-f} ($\overline{P}$ has such a subgroup), we have $|P_2| = p^e$, $P_2 \trianglelefteq P$ and P_2 is noncyclic. Then $P_2 \in \prod_{M_G^{*p}}(G)$. It is not very difficult to see that $\overline{P_2} \in \prod_{\overline{M}_{\overline{G}}^{*p}}(\overline{G})$. Hence the hypotheses are inherited by $\overline{M}$. By induction, we see that every p-chief factor of $\overline{G}$ below $\overline{M}$ is cyclic, and thus $M_G^{*p} \leq N$. Recall that $N \leq M_G^{*p}$. Then $M_G^{*p} = N$ is a minimal normal subgroup of G.

Step 4 $|M_G^{*p}| \geq p^2$.

Assume that $|M_G^{*p}| < p^2$. By Step 3, it follows that $|M_G^{*p}| = p$. Hence every p-chief factor of G below M is cyclic. This is a contradiction.

Step 5 $P \trianglelefteq G$.

Let $T/M_G^{*p} = O_{p'}(M/M_G^{*p})$, where $M_G^{*p} \leq T \leq M$. Let K be a Hall p'-subgroup of T. We claim that $K = 1$, i.e., $O_{p'}(M/M_G^{*p}) = 1$. Assume that $K > 1$ and we work to obtain a contradiction. By Step 1 and $K > 1$, we see that T is not p-nilpotent. Recall that M_G^{*p} is a minimal normal subgroup of G and M_G^{*p} is an elementary abelian p-subgroup (Step 3). By Lemma 2.6, it follows that $|M_G^{*p}| = p$. This contradicts to Step 4. Hence $O_{p'}(M/M_G^{*p}) = 1$. Note that M/M_G^{*p} is p-supersolvable. Hence M/M_G^{*p} is p-solvable with p-length 1. Since $O_{p'}(M/M_G^{*p}) = 1$, we see that $P/M_G^{*p} \trianglelefteq G/M_G^{*p}$, and thus $P \trianglelefteq G$.

Step 6 The final contradiction.

Since $M_G^{*p}, P \trianglelefteq G$(Step 5), $|M_G^{*p}| \leq p^e < p^{e+1} \leq |P|$ and every chief factor of G/M_G^{*p} below P/M_G^{*p} is cyclic, we see that P has a subgroup U with order p^{e+1} such that $M_G^{*p} < U \leq P$ and $U \trianglelefteq G$. It is not very difficult to see that $U_G^{*p} = P_G^{*p} = M_G^{*p}$.

We claim that $M_G^{*p} \cap \Phi(P) = 1$. Assume that $M_G^{*p} \cap \Phi(P) > 1$ and we work to obtain a contradiction. Since M_G^{*p} is a minimal normal subgroup of G, we see that $M_G^{*p} \leq \Phi(P)$. Since every chief factor of G/M_G^{*p} below P/M_G^{*p} is cyclic, by Corollary 3.28 of [5], P/M_G^{*p} is centralized by $O^p(G_p^*)$. By Corollary 3.29 of [5], we see that P is centralized by $O^p(G_p^*)$. By Lemma 2.8, we see that every chief factor of G below P is cyclic, and thus $M_G^{*p} = P_G^{*p} = 1$. This is a contradiction.

Let $S \in Syl_p(G)$. Then $P \leq S$. Note that $1 < M_G^{*p} \trianglelefteq S$. Then M_G^{*p} has a maximal subgroup N_1 such that $N_1 \trianglelefteq S$. By Lemma 2.7, P has a maximal subgroup P_1 such that $N_1 = P_1 \cap M_G^{*p}$. Note that $[U : U \cap P_1]$ divides p. It is not very difficult to see that $U \cap P_1$ is a maximal subgroup of U (otherwise, we have $U \cap P_1 = U$, and thus $P_1 \cap M_G^{*p} = (P_1 \cap U) \cap M_G^{*p} = M_G^{*p} > N_1$. This is a contradiction). Hence $U \cap P_1$ is a normal subgroup of P with order p^e and $(U \cap P_1) \cap M_G^{*p} = N_1$. If $U \cap P_1$ is noncyclic, then $U \cap P_1 \in \prod_{M_G^{*p}}(G)$. Hence $[G : N_G(N_1)]$ is a p-number, and thus $G = N_G(N_1)S$. Recall that $N_1 \trianglelefteq S$. Then $N_1 \trianglelefteq G$. Recall that M_G^{*p} is a minimal normal subgroup of G and N_1 is a maximal subgroup of M_G^{*p}. Then $N_1 = 1$ and $|M_G^{*p}| = p$. This contradicts to Step 4. If $U \cap P_1$ is cyclic, then U has a cyclic maximal subgroup. Since $e \geq 2$, we see that $|U| = p^{e+1} \geq p^3$. By Step 4, it follows that $U_G^{*p} = M_G^{*p}$ is an elementary abelian p-subgroup with order exceeding p. Note that Q_8 has exactly one subgroup with order 2. Hence U is neither elementary abelian of order p^2 nor isomorphic to Q_8. By Lemma 2.1, we see that U is centralized by $O^p(G_p^*)$. By Lemma 2.8, we see that every chief factor of G below U is cyclic, and thus $M_G^{*p} = U_G^{*p} = 1$. This is the final contradiction. □

Theorem 1.12 has the following three corollaries.

Corollary 3.11. *Let G be a finite group, $M \trianglelefteq G$, p be a prime divisor of $|M|$ and $P \in Syl_p(M)$. Suppose that for any maximal subgroup P_1 of P, $P_1 \in \prod_{M_G^{*p}}(G)$. If $P \cap M_G^{*p} < P$, then every p-chief factor of G below M is cyclic.*

Corollary 3.12. *Let G be a finite group, $M \trianglelefteq G$, p be a prime divisor of $|M|$, e be an integer, and $P \in Syl_p(M)$ with $|P| \geq p^{e+1}$. Suppose that for any normal subgroup P_1 of P with order p^e, $P_1 \in \prod_{M_G^{*p}}(G)$. If $|P \cap M_G^{*p}| \leq p^e$, then every p-chief factor of G below M is cyclic.*

Corollary 3.13. *Let G be a finite group, $M \trianglelefteq G$, p be a prime divisor of $|M|$, $e \geq 2$ be an integer, and $P \in Syl_p(M)$ with $|P| \geq p^{e+1}$. Suppose that for any normal noncyclic subgroup P_1 of P with order p^e (if P has such a subgroup), $P_1 \in \prod_{M_G^{*p}}(G)$. If $|P \cap M_G^{*p}| \leq p^e$, then every p-chief factor of G below M is cyclic.*

Now we work to prove Theorem 1.13–1.17.

Proof of Theorem 1.13. By Theorem 3.1, it follows that every 2-chief factor of G below X is cyclic. Hence every 2-chief factor of G below $F_2^*(M)$ is cyclic. In particular, $F_2^*(M)$ is 2-nilpotent. Recall that $O_{2'}(M) \leq F_2^*(M)$ and $F_2^*(M)/O_{2'}(M) = F^*(M/O_{2'}(M))$. It is not very difficult to see that $F^*(M/O_{2'}(M))$ is a 2-subgroup. Then every chief factor of $G/O_{2'}(M)$ below $F^*(M/O_{2'}(M))$ is cyclic. By Lemma 2.10, it follows that every chief factor of $G/O_{2'}(M)$ below $M/O_{2'}(M)$ is cyclic. This completes the proof. □

Proof of Theorem 1.14. By Theorem 3.3, it follows that every p-chief factor of G below X is cyclic. Hence every p-chief factor of G below $F_p^*(M)$ is cyclic. In particular, $F_p^*(M)$ is p-supersovable. Recall that $O_{p'}(M) \leq F_p^*(M)$ and $F_p^*(M)/O_{p'}(M) = F^*(M/O_{p'}(M))$. Hence every p-chief factor of $G/O_{p'}(M)$ below $F^*(M/O_{p'}(M))$ is cyclic. Since $M/O_{p'}(M)$ is a normal subgroup of $G/O_{p'}(M)$, $F^*(M/O_{p'}(M))$ is p-solvable, $O_{p'}(M/O_{p'}(M)) = 1$ and every p-chief factor of $G/O_{p'}(M)$ below $F^*(M/O_{p'}(M))$ is cyclic, by Lemma 2.11, it follows that every chief factor of $G/O_{p'}(M)$ below $M/O_{p'}(M)$ is cyclic. This completes the proof. □

Proof of Theorem 1.15. At first, we work to prove that every p-chief factor of G below X is cyclic. If $|P \cap X_G^{*p}| \leq p^e$ or $P \cap X_G^{*p}$ is cyclic, by Theorem 1.12, it follows that every p-chief factor of G below X is cyclic. If $|P \cap X_G^{*p}| \geq p^{e+1}$ and $P \cap X_G^{*p}$ is noncyclic, by Theorem 3.8, we see that every p-chief factor of G below X_G^{*p} is cyclic, and thus every p-chief factor of G below X is cyclic.

Using the same arguments in the proof of Theorem 1.14, it follows that every chief factor of $G/O_{p'}(M)$ below $M/O_{p'}(M)$ is cyclic. □

Proof of Theorem 1.16. At first, we work to prove that every p-chief factor of G below X is cyclic. If $|P \cap X_G^{*p}| \leq p^2$ or $P \cap X_G^{*p}$ is cyclic, by Theorem 1.12, it follows that every p-chief factor of G below X is cyclic. If $|P \cap X_G^{*p}| \geq p^3$ and $P \cap X_G^{*p}$ is noncyclic, by Theorem 3.9, we see that every p-chief factor of G below X_G^{*p} is cyclic, and thus every p-chief factor of G below X is cyclic.

Using the same arguments in the proof of Theorem 1.14, it follows that every chief factor of $G/O_{p'}(M)$ below $M/O_{p'}(M)$ is cyclic. □

Proof of Theorem 1.17. At first, we work to prove that every p-chief factor of G below X is cyclic. If $P \cap X_G^{*p} = 1$, it is not very difficult to see that every p-chief factor of G below X is cyclic. If $P \cap X_G^{*p} > 1$ is cyclic, by Theorem 3.1 and Theorem 3.3, we see that every p-chief factor of G below X_G^{*p} is cyclic, and thus every p-chief factor of G below X is cyclic. If $P \cap X_G^{*p} > 1$ is noncyclic, by Theorem 3.10, we see that every p-chief factor of G below X_G^{*p} is cyclic, and thus every p-chief factor of G below X is cyclic.

Using the same arguments in the proof of Theorem 1.14, it follows that every chief factor of $G/O_{p'}(M)$ below $M/O_{p'}(M)$ is cyclic. □

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