A note on S-semipermutable subgroups of finite groups

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ABSTRACT – In this note, we obtain some criteria for p-supersolvability of a finite group and extend some known results concerning weakly S-semipermutable subgroups.


KEYWORDS. p-supersolvability, weakly S-semipermutable subgroups.

1. Introduction

Throughout the paper, we suppose G is a finite group and p is a prime. Let π(G) be the set of all the prime divisors of |G|. Let O_p(G) = \bigcap\{N : N \trianglelefteq G \text{ and } G/N \text{ is a } p\text{-group}\}. To state our results, we need to recall some notation. According to Kegel (see [6]), let H be a subgroup of a finite group G; then H is called an S-permutable subgroup of G if H permutes with every Sylow subgroup of G. According to Chen (see [2]), let H be a subgroup of a finite group G; then H is said to be S-semipermutable in G if HQ = QH for all Sylow q-subgroups Q of G for all primes q not dividing |H|. According to Li et al. (see [7]), let H be a subgroup of a finite group G; then H is called a weakly S-semipermutable subgroup of G if there exist T ≤ G and H_1 ≤ G such that G = HT, H \cap T ≤ H_1 ≤ H and H_1 is S-semipermutable in G. Following Yakov Berkovich and I. M. Isaacs (see [1]), if G is a finite group and p is a prime divisor of |G|, we write G^*_p to denote the unique smallest normal subgroup of G for which the corresponding factor group is abelian of exponent dividing p − 1. It is well known that G is p-supersolvable if and only if G^*_p is p-nilpotent (see Lemma 3.6 of [1]).

Recently, we proved the following theorem.

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Theorem 1.1 (Theorem 1.2, [9]). Let $p$ be a prime dividing the order of a finite group $G$, $e$ be a positive integer and $P \in \text{Syl}_p(G)$ with $|P| \geq p^{e+1}$. Then $G$ is $p$-supersolvable if and only if $|P \cap \text{O}^p(G^*_p)| \leq p^e$ and $P \cap \text{O}^p(G^*_p)$ is $S$-permutable in $G$ for all subgroups $P_1 \leq P$ with $|P_1| = p^e$.

In this note, at first, we generalize Theorem 1.2 of [9] as follows.

Theorem 1.2. Let $p$ be a prime dividing the order of a finite group $G$, $e$ be a positive integer, $P \in \text{Syl}_p(G)$ with $|P| \geq p^{e+1}$ and $L \leq G$ with $\text{O}^p(G^*_p) \leq L \leq G$. Suppose that $|P \cap L| \leq p^e$ and $P \cap L$ is $S$-permutable in $G$ for all subgroups $P_1 \leq P$ with $|P_1| = p^e$. Then $G$ is $p$-supersolvable.

Using Theorem 1.2, we prove the following results which generalize Theorem 1.3 and Theorem 1.4 of [9].

Theorem 1.3. Let $p$ be a prime dividing the order of a finite group $G$, $e \geq 2$ be an integer, $P \in \text{Syl}_p(G)$ with $|P| \geq p^{e+1}$ and $L \leq G$ with $\text{O}^p(G^*_p) \leq L \leq G$. Suppose that $P \cap L$ is $S$-permutable in $G$ for all subgroups $P_1 \leq P$ with $|P_1| = p^e$. Then $G$ is $p$-supersolvable.

Let $p$ be a prime and $P$ be a nonidentity $p$-group with $|P| = p^n$. We define the set $\Omega(P)$. If $p = 2$ and $P$ is non-abelian, let $\Omega(P) = \{P_1 | P_1 \leq P \text{ and } |P_1| = 2\} \cup \{P_2 | P_2 \leq P \text{ and } P_2 \text{ is a cyclic subgroup of order 4} \}$. Otherwise, let $\Omega(P) = \{P_1 | P_1 \leq P \text{ and } |P_1| = p\}$.

Theorem 1.4. Let $p$ be a prime dividing the order of a finite group $G$, $P \in \text{Syl}_p(G)$ and $L \leq G$ with $\text{O}^p(G^*_p) \leq L \leq G$. Suppose that $P \cap L$ is $S$-permutable in $G$ for all subgroups $P_1 \in \Omega(P)$. Then $G$ is $p$-supersolvable.

Note that Theorem 1.3 and Theorem 1.4 also generalize Theorem 3.5 of [7] and Theorem 3.1, Theorem 3.4 of [8].

2. Preliminaries

Lemma 2.1. Let $p$ be a prime dividing the order of a finite group $G$. Then the following results hold.

(a) (Lemma 3.1, [1]) Let $P_1 \leq G$ be a $p$-group and $N \leq G$. If $P_1$ is $S$-permutable in $G$, then $P_1N/N$ is $S$-permutable in $G/N$.

(b) (Lemma 3.2, [1]) Let $P_1 \leq G$ be a $p$-group and $N$ be a normal $p$-subgroup of $G$. If $P_1$ is $S$-permutable in $G$, then $P_1 \cap N$ is normalized by $\text{O}^p(G)$. In particular, if $P_1 \leq N$, then $P_1$ is normalized by $\text{O}^p(G)$.

(c) (Lemma 3.3, [1]) Let $X \leq H \leq G$. If $X$ is $S$-permutable in $G$, then $X$ is $S$-permutable in $H$. 

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Lemma 2.2 (Lemma 2.1, [9]). Let $p$ be a prime dividing the order of a finite group $G$, $P \in \text{Syl}_p(G)$, $N \trianglelefteq G$ and let $e$ be a positive integer. Write $P_1 = P \cap N$. Assume that $P_1 \trianglelefteq N$ and $N$ is not $p$-nilpotent. Also assume that $|P_1| \leq p^e$ and $|P| \geq p^{e+1}$. Then $P$ has a normal subgroup $P_2$ of order $p^e$ with $[P_1 : P_1 \cap P_2] = p$.

Lemma 2.3 (Lemma 2.2, [9]). Let $p$ be a prime dividing the order of a finite group $G$ and $P \in \text{Syl}_p(G)$. Write $\hat{P} = P \cap O^p(G_2^0)$. Assume that $\hat{P} > 1$ and $\hat{P}$ has a maximal subgroup $T$ with $T \trianglelefteq G$. Then $\hat{P} \not\trianglelefteq G$.

Lemma 2.4 (Lemma 2.8, [9]). Let $p$ be a prime dividing the order of a finite group $G$ and $P_1$ be a $p$-subgroup of $G$. Let $L \subseteq G$ and $N$ be a normal $p'$-subgroup of $G$. Then $P_1 N / N \cap LN / N = (P_1 \cap L) N / N$.

Lemma 2.5 (Lemma 2.9, [9]). Let $p$ be a prime dividing the order of a finite group $G$ and $N \trianglelefteq G$. Then $(G / N)_p^* = G_p^* N / N$, $O^p(G / N) = O^p(G) N / N$ and $O^p((G / N)_p^*) = O^p(G_p^* N) / N$.

Recently, I. M. Isaacs proved the following significant theorem.

Lemma 2.6 (Theorem A, [5]). Let $p$ be a prime dividing the order of a finite group $G$ and $P_1$ be an $S$-semipermutable $p$-subgroup of $G$. Then $P_1^G$ is solvable.

Recently, Yakov Berkovich and I. M. Isaacs proved the following powerful results.

Lemma 2.7 (Yakov Berkovich and I. M. Isaacs). Let $p$ be a prime and $P$ be a nonidentity finite $p$-group. Let $A$ act on $P$ via automorphisms.

- (a) (Lemma 2.1(a), [1]) If $P$ is cyclic, then $O^p(A_1^*)$ acts trivially on $P$.
- (b) (Theorem A, [1]) Fix an integer $e \geq 3$. If $P$ is a noncyclic $p$-group with $|P| \geq p^{e+1}$ and every noncyclic subgroup of $P$ with order $p^e$ is stabilized by $O^p(A)$, then $O^p(A_1^*)$ acts trivially on $P$.
- (c) (Corollary B, [1]) If $P$ is a noncyclic $p$-group with $|P| \geq p^3$ and every subgroup of $P$ with order $p^2$ is stabilized by $O^p(A)$, then $O^p(A_1^*)$ acts trivially on $P$.

Lemma 2.8 (Yakov Berkovich and I. M. Isaacs). Let $p$ be a prime dividing the order of a finite group $G$ and $P \in \text{Syl}_p(G)$.

- (a) (Lemma 3.8, [1]) If $P$ is cyclic and some nonidentity subgroup $U \leq P$ is $S$-semipermutable in $G$, then $G$ is $p$-supersolvable.
- (b) (Theorem D, [1]) Fix an integer $e \geq 3$. If $P$ is a noncyclic $p$-group with $|P| \geq p^{e+1}$ and every noncyclic subgroup of $P$ with order $p^e$ is $S$-semipermutable in $G$, then $G$ is $p$-supersolvable.
- (c) (Corollary E, [1]) If $P$ is a noncyclic $p$-group with $|P| \geq p^3$ and every subgroup of $P$ with order $p^2$ is $S$-semipermutable in $G$, then $G$ is $p$-supersolvable.
3. Main Results

Proof of Theorem 1.2. Suppose that $G$ is a counterexample with minimal order; we complete the following steps to obtain a contradiction. Since $G$ is not $p$-supersolvable, it follows that $O^p(G_p^*)$ is not $p$-nilpotent.

Step 1. $P \cap L \geq P \cap O^p(G_p^*) > 1$.

Since $O^p(G_p^*)$ is not $p$-nilpotent, it follows that $P \cap O^p(G_p^*) > 1$. Since $L \geq O^p(G_p^*)$, it follows that $P \cap L \geq P \cap O^p(G_p^*) > 1$.

Step 2. $O_p(G) = 1$.

By Lemma 2.1(a), Lemma 2.4 and Lemma 2.5, the hypotheses are inherited by $G/O_p(G)$. If $O_p(G) > 1$, then $G/O_p(G)$ is $p$-supersolvable, and thus $G$ is $p$-supersolvable. This is a contradiction. Hence $O_p(G) = 1$.

Step 3. Let $\bar{P} = P \cap O^p(G_p^*)$. Then $\bar{P} \leq G$.

Let $U = P \cap L$. Then $U \leq P$ and $|U| \leq p^\varepsilon$. Hence $P$ has a normal subgroup $P_1$ of order $p^\varepsilon$ with $U \leq P_1$. By the hypotheses, $U = P_1 \cap L$ is $S$-semipermutable in $G$. By Lemma 2.6, $U^G$ is solvable. By Step 1, $U^G \geq U > 1$. By Step 2, $O_p(U^G) = 1$. Since $U^G$ is solvable, it follows that $O_p(U^G) > 1$. Let $N = O_p(U^G)$. Since $U^G \leq L$, it follows that $N \leq U$. Hence $|N| \leq |U| \leq p^\varepsilon$.

Assume that $1 < |N| < p^\varepsilon$. Recall that $N \leq P \cap L$. By Lemma 2.1(a) and Lemma 2.5, the hypotheses are inherited by $G/N$. Hence $G/N$ is $p$-supersolvable. By Lemma 2.5, it follows that $O^p(G_p^*)N/N = O^p((G/N)_p^*)$ is a $p'$-group, and thus $N \cap O^p(G_p^*)$ is the normal Sylow $p$-subgroup of $O^p(G_p^*)$. Hence $\bar{P} = N \cap O^p(G_p^*)$, and thus $\bar{P} \leq G$.

Assume that $|N| = p^\varepsilon$. Since $N \leq U$ and $|U| \leq p^\varepsilon$, it follows that $U = N$ is a normal subgroup of $G$. Hence $\bar{P} = P \cap O^p(G_p^*) = U \cap O^p(G_p^*) \leq G$.

Step 4. The final contradiction.

Recall that $\bar{P}$ is the normal Sylow $p$-subgroup of $O^p(G_p^*)$ (Step 3), $O^p(G_p^*)$ is not $p$-nilpotent, $|P| > p^{\varepsilon + 1}$ and $1 < |\bar{P}| \leq |U| \leq p^\varepsilon$. By Lemma 2.2, $P$ has a normal subgroup $P_2$ of order $p^\varepsilon$ with $[\bar{P} : \bar{P} \cap P_2] = p$. Note that $\bar{P} \cap P_2 \leq P$. By the hypotheses, $P_2 \cap L$ is $S$-semipermutable in $G$. By Lemma 2.1(b), $\bar{P} \cap P_2 = \bar{P} \cap P_2 \cap L$ is normalized by $O_p(G)$. Hence $\bar{P} \cap P_2 \leq G$. By Lemma 2.3, $\bar{P} \not\leq G$. This is a contradiction since $\bar{P} \leq G$. Hence we obtain the final contradiction.

Proof of Theorem 1.3. We proceed by induction on $|G|$. By Lemma 2.1(a), Lemma 2.4 and Lemma 2.5, the hypotheses are inherited by $G/O_p(G)$. If $O_p(G) > 1$, by induction, $G/O_p(G)$ is $p$-supersolvable, and thus $G$ is $p$-supersolvable. So we can assume $O_p(G) = 1$. Let $U = P \cap L$. If $|U| \leq p^\varepsilon$, by Theorem 1.2, $G$ is $p$-supersolvable. Assume that $|U| > p^{\varepsilon + 1}$. For any subgroup $P_1 \leq U$ with $|P_1| = p^\varepsilon$, $P_1$ is $S$-semipermutable in $G$. By Lemma 2.1(c), $P_1$ is $S$-semipermutable in $L$. By
Lemma 2.8, \(L\) is \(p\)-supersolvable, and thus \(L\) is \(p\)-solvable with \(p\)-length 1. Since \(O_{p'}(G) = 1\), it follows that \(U\) is the normal Sylow \(p\)-subgroup of \(L\), and thus \(U \leq G\). Note that for all subgroups \(P_1 \leq U\) with \(|P_1| = p^e\), \(P_1\) is \(S\)-semipermutable in \(G\). By Lemma 2.1(b), it follows that \(P_1\) is normalized by \(O_P(G)\). By Lemma 2.7, \(U\) is centralized by \(O^p(G_\ast^p)\). Let \(\hat{P} = P \cap O^p(G_\ast^p)\). Note that \(\hat{P} = U \cap O^p(G_\ast^p)\), and thus \(\hat{P} \leq Z(O^p(G_\ast^p))\). By Burnside’s Theorem (see Theorem 5.13 of [4]), \(O^p(G_\ast^p)\) is \(p\)-nilpotent, i.e., \(G_p\) is \(p\)-nilpotent. Hence \(G\) is \(p\)-supersolvable. □

**Proof of Theorem 1.4.** By Lemma 2.1(a), Lemma 2.4 and Lemma 2.5, it is no loss to assume that \(O_{p'}(G) = 1\). Assume \(G\) is not \(p\)-supersolvable; we work to obtain a contradiction. Since \(G\) is not \(p\)-supersolvable, it follows that \(O^p(G_\ast^p)\) is not \(p\)-nilpotent. Let \(\hat{P} = P \cap O^p(G_\ast^p)\). Then \(\hat{P} > 1\). Since \(O^p(G_\ast^p) \leq L\), it follows that for any \(P_1 \in \Omega(\hat{P})\), \(P_1\) is \(S\)-semipermutable in \(G\). By Lemma 2.6, \(P_1^G\) is solvable. Let \(M = \prod_{P_1 \in \Omega(\hat{P})} P_1^G\). Then \(M\) is a solvable normal subgroup of \(G\) and \(M \leq O^p(G_\ast^p)\). Since \(\hat{P} > 1\) and for any \(P_1 \in \Omega(\hat{P})\), \(P_1 \leq M\), it follows that \(M > 1\). Since \(O_{p'}(G) = 1\) and \(M \leq G\), we have \(O_{p'}(M) = 1\). Since \(M > 1\) is solvable and \(O_{p'}(M) = 1\), we have \(O_{p'}(M) > 1\). Note that \(O_p(M) \leq P \cap L\). Hence for any \(P_2 \in \Omega(O_p(M))\), \(P_2\) is \(S\)-semipermutable in \(G\). By Lemma 2.1(b), \(P_2\) is normalized by \(O^p(G)\). By Lemma 2.9, \(O_p(M)\) is centralized by \(O^p(G_\ast^p)\). Recall that \(M \leq O^p(G_\ast^p)\). Hence \(O_p(M) \leq Z(M)\). Since \(M\) is solvable and \(O_{p'}(M) = 1\), by Hall-Higman’s Lemma (see Theorem 3.21 of [4]), it follows that \(M = O_p(M)\). Hence \(M\) is centralized by \(O^p(G_\ast^p)\). Recall that for any \(P_1 \in \Omega(\hat{P})\), \(P_1 \leq M\). Hence for any \(P_1 \in \Omega(\hat{P})\), \(P_1\) is centralized by \(O^p(G_\ast^p)\). By Satz IV.5.5 of [3], \(O^p(G_\ast^p)\) is \(p\)-nilpotent. This is the desired contradiction. Hence \(G\) is \(p\)-supersolvable. □

4. Final Remarks

Theorem 1.3 and Theorem 1.4 have the following corollaries.

**Corollary 4.1.** Let \(p\) be a prime dividing the order of a finite group \(G\), \(e \geq 2\) be an integer and \(P \in Syl_p(G)\) with \(|P| \geq p^{e+1}\). Suppose that \(P_1 \cap O^p(G)\) is \(S\)-semipermutable in \(G\) for all subgroups \(P_1 \leq P\) with \(|P_1| = p^e\). Then \(G\) is \(p\)-supersolvable.

**Corollary 4.2.** Let \(p\) be a prime dividing the order of a finite group \(G\) and \(P \in Syl_p(G)\). Suppose that \(P_1 \cap O^p(G)\) is \(S\)-semipermutable in \(G\) for all subgroups \(P_1 \in \Omega(P)\). Then \(G\) is \(p\)-supersolvable.

Note that Corollary 4.1 and Corollary 4.2 generalize Theorem 3.5 of [7].

**Remark 4.3.** For any odd prime \(p\) and any positive integer \(e\), there exists a finite group \(G\) with \(p\) an odd prime divisor of \(|G|\), \(P \in Syl_p(G)\) and
$|P| \geq p^{e+1}$ such that for every subgroup $P_1$ of $P$ with order $p^e$, $P_1 \cap O^p(G)$ is $S$-semipermutable in $G$, but $P$ has a subgroup $P_3$ of order $p^e$ such that $P_3$ is not weakly $S$-semipermutable in $G$. Hence our Corollary 4.1 and Corollary 4.2 are stronger than Theorem 3.5 of [7]. See the following example.

Example 4.4. Let $n$ be an integer with $n > e$, $p$ be an odd prime and $T = \langle a, b \mid a^{p^n} = b^2 = 1, b^{-1}ab = a^{-1} \rangle \cong D_{2p^n}$. There exists $c \in Aut(T)$ such that $a^c = a$ and $b^c = ba$. Consider $G = T \times \langle c \rangle \cong \langle a, b, c \mid a^{p^n} = b^2 = c^p = 1, b^{-1}ab = a^{-1}, c^{-1}ac = a, c^{-1}bc = ba \rangle$.

Let $P = \langle a \rangle \times \langle c \rangle$. Then $P$ is the normal Sylow $p$-subgroup of $G$ with order $p^{2n}$. Then $b \in O^p(G)$, so $ba = b^c \in O^p(G)$. Hence $a \in O^p(G)$, so $O^p(G) = T$. Hence for any subgroup $P_1$ of $P$ with order $p^e$, $P_1 \cap O^p(G)$ is normal in $G$, and thus $S$-semipermutable in $G$. Let $\bar{c} = c^{p^n-e}$. Consider $\langle \bar{c} \rangle$. Note that $|\langle \bar{c} \rangle| = p^e$. Then $\langle \bar{c} \rangle$ is not weakly $S$-semipermutable in $G$. To see this, assume that $\langle \bar{c} \rangle$ is weakly $S$-semipermutable in $G$, since $\langle \bar{c} \rangle \leq \Phi(\langle \bar{c} \rangle)$, it follows that $\langle \bar{c} \rangle$ is $S$-semipermutable in $G$. Hence $\langle \bar{c} \rangle$ normalizes $b$. This is a contradiction since $\bar{c}^{-1}b\bar{c} = ba^{p^n-e}$. Hence $\langle \bar{c} \rangle$ is not weakly $S$-semipermutable in $G$.

References


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