Rend. Sem. Mat. Univ. Padova 1xx (201x) **Rendiconti del Seminario Matematico della Università di Padova**

The flatness of ternary cyclotomic polynomials

Bin Zhang*

Abstract – It is well known that all of the prime cyclotomic polynomials and binary cyclotomic polynomials are flat, and the flatness of ternary cyclotomic polynomials is much more complicated. Let $p < q < r$ be odd primes such that $zr \equiv \pm 1 \pmod{pq}$, where z is a positive integer. So far, the classification of flat ternary cyclotomic polynomials for $1 \leq z \leq 5$ has been given. In this paper, for $z = 6$ and $q \equiv \pm 1 \pmod{p}$, we give the necessary and sufficient conditions for ternary cyclotomic polynomials $\Phi_{pqr}(x)$ to be flat.

Mathematics Subject Classification (2010). 11B83, 11C08, 11N56.

Keywords. Flat cyclotomic polynomial; Ternary cyclotomic polynomial; Coefficients of cyclotomic polynomial.

1. Introduction

The n-th cyclotomic polynomial is the monic polynomial whose roots are the primitive nth roots of unity, that is

$$\Phi_n(x) = \prod_{\substack{1 \leq k \leq n \\ (k,n)=1}} (x - e^{\frac{2\pi ik}{n}}) = \sum_{j=0}^{\phi(n)} a(n,j)x^j,$$

where ϕ is the Euler totient function. Let

$$A(n) = \max\{|a(n,j)| : 0 \leq j \leq \phi(n)\}.$$

*The author was supported by National Natural Science Foundation of China (Grant No. 11801303), project ZR2019QA016 supported by Shandong Provincial Natural Science Foundation, Project funded by China Postdoctoral Science Foundation (Grant No. 2018M640617) and Science and Technology Project of Qufu Normal University (Grant No. xkj201605).

Bin Zhang, School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China
E-mail: zhangbin100902025@163.com

We say that a cyclotomic polynomial is $flat$ if $A(n) = 1$. To determine $A(n)$, one can restrict to the case when n is a product of distinct odd primes, because all other cases can be trivially reduced to it.

Throughout the paper, we assume that $p < q < r$ are odd primes. It is clear that all the cyclotomic polynomials $\Phi_p(x)$ and $\Phi_{pq}(x)$ are flat. It turns out that the flatness of ternary cyclotomic polynomials $\Phi_{pqr}(x)$ becomes much more complicated. This topic has been considered by several authors, see, for example, [1, 2, 3, 4, 5, 6, 10, 11, 12, 13, 14], but a complete classification of flat ternary cyclotomic polynomials is not known.

Based on numerical data, Elder [3] presented the following conjecture in 2012.

Conjecture 1.1. *Let $p < q < r$ be odd primes such that $A(pqr) = 1$. Then either $q \equiv \pm 1 \pmod{p}$ or $r \equiv \pm 1 \pmod{pq}$.*

Let
$$zr \equiv \pm 1 \pmod{pq},$$
where z is an integer. In 2007, Kaplan [6] showed that if $z = 1$, then $A(pqr) = 1$. On invoking Conjecture 1.1, it seems likely that if $A(pqr) = 1$ and $z \neq 1$, then we certainly have $q \equiv \pm 1 \pmod{p}$. So we are interested in the flatness of ternary cyclotomic polynomials when $q \equiv \pm 1 \pmod{p}$ and $z \neq 1$.

At present, the flatness of $\Phi_{pqr}(x)$ in the cases $1 \leq z \leq 5$ have been studied by [3, 5, 6, 10, 11, 14], and it is well understood. In this paper, we consider the coefficients of ternary cyclotomic polynomials $\Phi_{pqr}(x)$ in the cases $q \equiv \pm 1 \pmod{p}$ and $z = 6$, and prove the following results.

Theorem 1.2. *Let $p < q < r$ be odd primes such that $q \equiv \pm 1 \pmod{p}$ and $6r \equiv \pm 1 \pmod{pq}$. Then $\Phi_{pqr}(x)$ is flat if and only if one of the following two conditions hold:*
(a) *$p = 5$, $q \geq 29$ and $q \equiv -1 \pmod{5}$;*
(b) *$p = 7$, $q \geq 43$ and $q \equiv 1 \pmod{7}$.*

2. Preliminaries

We first introduce the following two results due to Kaplan [6] which are useful to prove our theorem.

Lemma 2.1. ([6]) *Let $p < q < r$ be odd primes. Then for any prime $s > q$ such that $s \equiv \pm r \pmod{pq}$, $A(pqr) = A(pqs)$.*

Lemma 2.2. ([6]) *Let $p < q < r$ be odd primes. Let $n \geq 0$ be an integer and $f(i)$ the unique value $0 \leq f(i) \leq pq - 1$ such that*

$$rf(i) + i \equiv n \pmod{pq}. \tag{2.1}$$

(1) *Then*

$$\sum_{i=0}^{p-1} a(pq, f(i)) = \sum_{i=0}^{p-1} a(pq, f(q+i)).$$

(2) *Set*

$$a^*(pq, i) = \begin{cases} a(pq, i) & \text{if } ri \le n; \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$a(pqr, n) = \sum_{i=0}^{p-1} a^*(pq, f(i)) - \sum_{i=0}^{p-1} a^*(pq, f(q+i)).$$

This technical lemma reduces the computation of the coefficients of $\Phi_{pqr}(x)$ to that of $\Phi_{pq}(x)$. Recall that these binary cyclotomic polynomial coefficients can be computed explicitly in the following lemma, see e.g. Lam and Leung [7] or Thangadurai [9].

Lemma 2.3. *Let $p < q$ be odd primes. Let s and t be positive integers such that $pq + 1 = ps + qt$ written uniquely. Then*

$$a(pq, j) = \begin{cases} 1 & \text{if } j = up + vq \text{ for some } 0 \le u \le s-1, 0 \le v \le t-1; \\ -1 & \text{if } j = up + vq + 1 \text{ for some } 0 \le u \le q-s-1, 0 \le v \le p-t-1; \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence of Lemma 2.3, we obtain the following two results in the cases $q \equiv \pm 1 \pmod{p}$.

Lemma 2.4. *Let $p < q$ be odd primes with $q = kp - 1$. Then*

$$a(pq, j) = \begin{cases} 1 & \text{if } j = up + vq \text{ for some } 0 \le u \le k-1, 0 \le v \le p-2; \\ -1 & \text{if } j = up + 1 \text{ for some } 0 \le u \le q-k-1; \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.5. *Let $p < q$ be odd primes with $q = kp + 1$. Then*

$$a(pq, j) = \begin{cases} 1 & \text{if } j = up \text{ for some } 0 \le u \le s-1; \\ -1 & \text{if } j = up + vq + 1 \text{ for some } 0 \le u \le k-1, 0 \le v \le p-2; \\ 0 & \text{otherwise.} \end{cases}$$

3. Auxiliary results for Theorem 1.2

By Lemma 2.1, it suffices to consider the case $6r \equiv 1 \pmod{pq}$ and $q \equiv \pm 1 \pmod{p}$ to prove Theorem 1.2. In this section, we give the following seven propositions about the coefficients of $\Phi_{pqr}(x)$. The proofs of these propositions will be given in Secs. 4, 5 and 6. Theorem 1.2 will be proved by showing these propositions.

3.1 $q \equiv -1 \pmod p$.

Proposition 3.1. *Let* $5 < q < r$ *be primes such that* $q \equiv -1 \pmod 5$ *and* $6r \equiv 1 \pmod{5q}$. *Then*

$$A(5qr) = \begin{cases} 2 & \text{if } q = 19; \\ 1 & \text{otherwise.} \end{cases}$$

Proposition 3.2. *Let* $7 < q < r$ *be primes such that* $q \equiv -1 \pmod 7$ *and* $6r \equiv 1 \pmod{7q}$.

(1) *If* $q = 13$, *then* $A(7 \cdot 13 \cdot r) = 2$.

(2) *If* $q > 13$, *then* $a(7qr, 4qr + 5) = 2$.

Proposition 3.3. *Let* $11 < q < r$ *be primes such that* $q \equiv -1 \pmod{11}$ *and* $6r \equiv 1 \pmod{11q}$. *Then* $a(11qr, 5qr + 33r + 1) = 2$.

Proposition 3.4. *Let* $11 < p < q < r$ *be primes such that* $q \equiv -1 \pmod p$ *and* $6r \equiv 1 \pmod{pq}$.

(1) *If* $p \equiv 1 \pmod 6$, *then* $a(pqr, 4qr + q + \frac{5p-5}{6}) = -2$.

(2) *If* $p \equiv 5 \pmod 6$, *then* $a(pqr, 3pr + 5qr + q + \frac{p-5}{6}) = -2$.

3.2 $q \equiv 1 \pmod p$.

Proposition 3.5. *Let* $5 < q < r$ *be primes such that* $q \equiv 1 \pmod 5$ *and* $6r \equiv 1 \pmod{5q}$. *Then* $a(5qr, qr + q + r + 3) = 2$.

Proposition 3.6. *Let* $7 < q < r$ *be primes such that* $q \equiv 1 \pmod 7$ *and* $6r \equiv 1 \pmod{7q}$. *Then*

$$A(7qr) = \begin{cases} 2 & \text{if } q = 29; \\ 1 & \text{otherwise.} \end{cases}$$

Proposition 3.7. *Let* $7 < p < q < r$ *be odd primes such that* $q \equiv 1 \pmod p$ *and* $6r \equiv 1 \pmod{pq}$.

(1) *If* $p \equiv 1 \pmod 6$, *then* $a(pqr, pqr + 3pr - 7qr + q + r + \frac{p-7}{6}) = 2$.

(2) *If* $p \equiv 5 \pmod 6$, $q = 2p + 1$, *then* $a(pqr, pqr - 9qr + q + r + \frac{5p-7}{6}) = 2$.

(3) *If* $p \equiv 5 \pmod 6$, $q \neq 2p + 1$, *then* $a(pqr, pqr - 7qr + q + r + \frac{5p-7}{6}) = 2$.

4. Proof of Proposition 3.1

By using the PARI/GP system, we have $A(5\cdot 19\cdot 491)=2$. Then, by Lemma 2.1, we get $A(5\cdot 19\cdot r)=2$ for those primes r with $6r\equiv 6\cdot 491\equiv \pm 1 \pmod{5\cdot 19}$. Next we show that $A(5qr)=1$, where

$$q\equiv -1 \pmod{p},\ q>19 \text{ and } 6r\equiv 1 \pmod{5q}.$$

Note that Lemma 2.2 yields

$$a(5qr,n)=\sum_{i=0}^{4} a^*(5q,f(i))+\sum_{i=q}^{q+4}\Big(-a^*(5q,f(i))\Big), \tag{4.1}$$

where $f(i)\equiv r^{-1}(n-i) \pmod{5q}$, $0\le f(i)\le 5q-1$, and

$$a^*(5q,f(i))=\begin{cases} a(5q,f(i)) & \text{if } rf(i)\le n;\\ 0 & \text{otherwise.}\end{cases} \tag{4.2}$$

For $5q+1=5s+qt$, we have $s=\frac{q+1}{5}$ and $t=4$. Thus in this case, we may rewrite the conclusion of Lemma 2.3 in the following form

$$a(5q,j)=\begin{cases} 1 & \text{if } j\equiv 0 \pmod 5 \text{ and } 0\le j\le q-4;\\ 1 & \text{if } j\equiv 4 \pmod 5 \text{ and } q\le j\le 2q-4;\\ 1 & \text{if } j\equiv 3 \pmod 5 \text{ and } 2q\le j\le 3q-4;\\ 1 & \text{if } j\equiv 2 \pmod 5 \text{ and } 3q\le j\le 4q-4;\\ -1 & \text{if } j\equiv 1 \pmod 5 \text{ and } 1\le j\le 4q-5;\\ 0 & \text{otherwise.}\end{cases} \tag{4.3}$$

For any given $n\in[0,\phi(5qr)]$, the value of $f(i)$ is uniquely defined and, by $rf(i)+i\equiv n \pmod{5q}$, we have

$$f(q)\equiv f(0)-q \pmod{5q}, \tag{4.4}$$

$$f(1)\equiv f(0)-6 \pmod{5q},\qquad f(q+1)\equiv f(0)-q-6 \pmod{5q}, \tag{4.5}$$

$$f(2)\equiv f(0)-12 \pmod{5q},\qquad f(q+2)\equiv f(0)-q-12 \pmod{5q}, \tag{4.6}$$

$$f(3)\equiv f(0)-18 \pmod{5q},\qquad f(q+3)\equiv f(0)-q-18 \pmod{5q}, \tag{4.7}$$

$$f(4)\equiv f(0)-24 \pmod{5q},\qquad f(q+4)\equiv f(0)-q-24 \pmod{5q}. \tag{4.8}$$

Let

$$S=\{0,1,2,3,4,q,q+1,q+2,q+3,q+4\}.$$

In order to use (4.1) and (4.2), we need to determine for which $i\in S$, $rf(i)\le n$. Now according to the value of $f(0)$, we give the following tables. The second row of each table is the inequality about $rf(i)$ for $i\in S$. In this section, for simplicity, we set

$$a_{f(i)}:=a(5q,f(i))$$

and let $\overline{f(0)}$ be the unique integer such that $0\le \overline{f(0)}\le 4$ and $\overline{f(0)}\equiv f(0) \pmod 5$. The values of $a_{f(i)}$ are obtained by using (4.3)–(4.8).

Table 1. $0 \leq f(0) \leq 5$

	$rf(1) > rf(2) > rf(3) > rf(4) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4) > rf(0)$									
$f(0)$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$
0	0	0	0	0	0	0	0	0	-1	1
1	0	0	0	0	0	1	0	0	0	-1
2	0	0	0	0	0	-1	1	0	0	0
3	0	0	0	0	0	0	-1	1	0	0
4	0	0	0	0	0	0	0	-1	1	0
5	0	0	0	0	0	0	0	0	-1	1

Table 2. $6 \leq f(0) \leq 11$

	$rf(2) > rf(3) > rf(4) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4) > rf(0) > rf(1)$									
$f(0)$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$
6	0	0	0	0	0	0	0	0	-1	1
7	0	0	0	0	0	1	0	0	0	-1
8	0	0	0	0	0	-1	1	0	0	0
9	0	0	0	0	0	0	-1	1	0	0
10	0	0	0	0	0	0	0	-1	1	0
11	0	0	0	0	0	0	0	0	-1	1

Table 3. $12 \leq f(0) \leq 17$

	$rf(3) > rf(4) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4) > rf(0) > rf(1) > rf(2)$									
$f(0)$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$
12	0	0	0	0	0	0	0	0	-1	1
13	0	0	0	0	0	1	0	0	0	-1
14	0	0	0	0	0	-1	1	0	0	0
15	0	0	0	0	0	0	-1	1	0	0
16	0	0	0	0	0	0	0	-1	1	0
17	0	0	0	0	0	0	0	0	-1	1

Table 4. $18 \leq f(0) \leq 23$

	$rf(4) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4) > rf(0) > rf(1) > rf(2) > rf(3)$									
$f(0)$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$
18	0	0	0	0	0	0	0	0	-1	1
19	0	0	0	0	0	1	0	0	0	-1
20	0	0	0	0	0	-1	1	0	0	0
21	0	0	0	0	0	0	-1	1	0	0
22	0	0	0	0	0	0	0	-1	1	0
23	0	0	0	0	0	0	0	0	-1	1

Table 5. $24 \leq f(0) \leq q-1$

	$rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4)$									
$\overline{f(0)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$
0	0	0	0	0	0	1	0	0	0	-1
1	0	0	0	0	0	-1	1	0	0	-1
2	0	0	0	0	0	0	-1	1	0	0
3	0	0	0	0	0	0	0	-1	1	0
4	0	0	0	0	0	0	0	0	-1	1

Table 6. $q \leq f(0) \leq q+5$

	$rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(q)$									
$f(0)$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$
q	0	0	0	0	1	0	0	-1	1	-1
$q+1$	0	0	0	0	0	0	0	0	-1	1
$q+2$	0	0	0	0	-1	1	0	0	0	0
$q+3$	0	0	0	0	0	-1	1	0	0	0
$q+4$	0	0	0	0	0	0	-1	1	0	0
$q+5$	0	0	0	0	1	0	0	-1	1	-1

Table 7. $q+6 \leq f(0) \leq q+11$

	$rf(q+2) > rf(q+3) > rf(q+4) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(q) > rf(q+1)$									
$f(0)$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$
$q+6$	0	0	0	0	1	0	0	-1	1	-1
$q+7$	0	0	0	-1	0	0	0	0	0	1
$q+8$	0	0	0	0	-1	1	0	0	0	0
$q+9$	0	0	0	0	0	-1	1	0	0	0
$q+10$	0	0	0	1	0	0	-1	1	-1	0
$q+11$	0	0	0	0	1	0	0	-1	1	-1

Table 8. $q+12 \leq f(0) \leq q+17$

	$rf(q+3) > rf(q+4) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(q) > rf(q+1) > rf(q+2)$									
$f(0)$	$-a_{f(q+3)}$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$
$q+12$	0	0	-1	0	1	0	0	0	1	-1
$q+13$	0	0	0	-1	0	0	0	0	0	1
$q+14$	0	0	0	0	-1	1	0	0	0	0
$q+15$	0	0	1	0	0	-1	1	-1	0	0
$q+16$	0	0	0	1	0	0	-1	1	-1	0
$q+17$	0	0	-1	0	1	0	0	0	1	-1

Table 9. $q+18 \le f(0) \le q+23$

	$rf(q+4) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3)$									
$f(0)$	$-a_{f(q+4)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$
$q+18$	0	0	-1	0	1	0	0	0	1	-1
$q+19$	0	0	0	-1	0	0	0	0	0	1
$q+20$	0	1	0	0	-1	1	-1	0	0	0
$q+21$	0	0	1	0	0	-1	1	-1	0	0
$q+22$	0	-1	0	1	0	0	0	1	-1	0
$q+23$	0	0	-1	0	1	0	0	0	1	-1

Table 10. $q+24 \le f(0) \le 2q-1$

	$rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4)$									
$\overline{f(0)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$
0	0	1	0	0	-1	1	-1	0	0	0
1	-1	0	1	0	0	0	1	-1	0	0
2	0	-1	0	1	0	0	0	1	-1	0
3	0	0	-1	0	1	0	0	0	1	-1
4	1	0	0	-1	0	-1	0	0	0	1

Table 11. $2q \le f(0) \le 2q+23$

	$rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4)$									
$f(0)$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$
$2q$	1	0	-1	0	1	-1	0	0	1	-1
$2q+1$	0	0	0	-1	0	0	0	0	0	1
$2q+2$	0	1	0	0	-1	1	-1	0	0	0
$2q+3$	-1	0	1	0	0	0	1	-1	0	0
$2q+4$	0	-1	0	1	0	0	0	1	-1	0
$2q+5$	1	0	-1	0	1	-1	0	0	1	-1
$2q+6$	0	1	0	-1	0	0	-1	0	0	1
$2q+7$	0	0	0	0	-1	1	0	0	0	0
$2q+8$	-1	0	1	0	0	0	-1	0	0	1
$2q+9$	0	-1	0	1	0	0	0	1	-1	0
$2q+10$	1	0	-1	0	1	-1	0	0	1	-1
$2q+11$	0	1	0	-1	0	0	-1	0	0	1
$2q+12$	0	0	1	0	-1	1	0	-1	0	0
$2q+13$	-1	0	0	0	0	0	1	0	0	0
$2q+14$	0	-1	0	1	0	0	0	1	-1	0
$2q+15$	1	0	-1	0	1	-1	0	0	1	-1
$2q+16$	0	1	0	-1	0	0	-1	0	0	1
$2q+17$	0	0	1	0	-1	1	0	-1	0	0
$2q+18$	-1	0	0	1	0	0	1	0	-1	0
$2q+19$	0	-1	0	0	0	0	0	1	0	0
$2q+20$	1	0	-1	0	1	-1	0	0	1	-1
$2q+21$	0	1	0	-1	0	0	-1	0	0	1
$2q+22$	0	0	1	0	-1	1	0	-1	0	0
$2q+23$	-1	0	0	1	0	0	1	0	-1	0

Table 12. $2q+24 \le f(0) \le 3q-1$

	$rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4)$									
$\overline{f(0)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$
0	0	0	1	0	-1	1	0	-1	0	0
1	-1	0	0	1	0	0	1	0	-1	0
2	0	-1	0	0	1	0	0	1	0	-1
3	1	0	-1	0	0	-1	0	0	1	0
4	0	1	0	-1	0	0	-1	0	0	1

Table 13. $3q \le f(0) \le 3q+23$

	$rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4)$									
$f(0)$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$
$3q$	1	-1	0	0	1	-1	0	1	0	-1
$3q+1$	0	0	-1	0	0	0	0	0	1	0
$3q+2$	0	1	0	-1	0	0	-1	0	0	1
$3q+3$	0	0	1	0	-1	1	0	-1	0	0
$3q+4$	-1	0	0	1	0	0	1	0	-1	0
$3q+5$	1	-1	0	0	1	-1	0	1	0	-1
$3q+6$	0	1	-1	0	0	0	-1	0	1	0
$3q+7$	0	0	0	-1	0	0	0	0	0	1
$3q+8$	0	0	1	0	-1	1	0	-1	0	0
$3q+9$	-1	0	0	1	0	0	1	0	-1	0
$3q+10$	1	-1	0	0	1	-1	0	1	0	-1
$3q+11$	0	1	-1	0	0	0	-1	0	1	0
$3q+12$	0	0	1	-1	0	0	0	-1	0	1
$3q+13$	0	0	0	0	-1	1	0	0	0	0
$3q+14$	-1	0	0	1	0	0	1	0	-1	0
$3q+15$	1	-1	0	0	1	-1	0	1	0	-1
$3q+16$	0	1	-1	0	0	0	-1	0	1	0
$3q+17$	0	0	1	-1	0	0	0	-1	0	1
$3q+18$	0	0	0	1	-1	1	0	0	-1	0
$3q+19$	-1	0	0	0	0	0	1	0	0	0
$3q+20$	1	-1	0	0	1	-1	0	1	0	-1
$3q+21$	0	1	-1	0	0	0	-1	0	1	0
$3q+22$	0	0	1	-1	0	0	0	-1	0	1
$3q+23$	0	0	0	1	-1	1	0	0	-1	0

Table 14. $3q+24 \le f(0) \le 4q-1$										
	$rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4)$									
$\overline{f(0)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$
0	0	0	0	1	-1	1	0	0	-1	0
1	-1	0	0	0	1	0	1	0	0	-1
2	1	-1	0	0	0	-1	0	1	0	0
3	0	1	-1	0	0	0	-1	0	1	0
4	0	0	1	-1	0	0	0	-1	0	1

Table 15. $4q \le f(0) \le 4q+20$										
	$rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4)$									
$f(0)$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$
$4q$	0	0	0	0	1	-1	1	0	0	-1
$4q+1$	0	-1	0	0	0	0	0	1	0	0
$4q+2$	0	1	-1	0	0	0	1	0	-1	0
$4q+3$	0	0	1	-1	0	0	0	-1	0	1
$4q+4$	0	0	0	1	-1	1	0	0	-1	0
$4q+5$	0	0	0	0	1	-1	1	0	0	-1
$4q+6$	0	0	0	0	0	0	-1	1	0	0
$4q+7$	0	0	-1	0	0	0	0	0	1	0
$4q+8$	0	0	1	-1	0	0	0	-1	0	1
$4q+9$	0	0	0	1	-1	1	0	0	-1	0
$4q+10$	0	0	0	0	1	-1	1	0	0	-1
$4q+11$	0	0	0	0	0	0	-1	1	0	0
$4q+12$	0	0	0	0	0	0	0	-1	1	0
$4q+13$	0	0	0	-1	0	0	0	0	0	1
$4q+14$	0	0	0	1	-1	1	0	0	-1	0
$4q+15$	0	0	0	0	1	-1	1	0	0	-1
$4q+16$	0	0	0	0	0	0	-1	1	0	0
$4q+17$	0	0	0	0	0	0	0	-1	1	0
$4q+18$	0	0	0	0	0	0	0	0	-1	1
$4q+19$	0	0	0	0	-1	1	0	0	0	0
$4q+20$	0	0	0	0	1	-1	1	0	0	-1

Table 16. $4q+21 \le f(0) \le 5q-1$										
	$rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4)$									
$\overline{f(0)}$	$a_{f(0)}$	$a_{f(1)}$	$a_{f(2)}$	$a_{f(3)}$	$a_{f(4)}$	$-a_{f(q)}$	$-a_{f(q+1)}$	$-a_{f(q+2)}$	$-a_{f(q+3)}$	$-a_{f(q+4)}$
0	0	0	0	0	0	1	0	0	0	-1
1	0	0	0	0	0	-1	1	0	0	0
2	0	0	0	0	0	0	-1	1	0	0
3	0	0	0	0	0	0	0	-1	1	0
4	0	0	0	0	0	0	0	0	-1	1

Let n in the range $0 \le n \le \phi(5qr)$ be fixed.

If $rf(i) > n$ holds for all $i \in S$, then $a^*(pq, f(i)) = 0$. So, by Lemma 2.2 (2), we infer that $a(5qr, n) = 0$.

If $rf(i) \le n$ holds for all $i \in S$, then $a^*(pq, f(i)) = a(pq, f(i))$. So, by Lemma 2.2 (1) and (2), we also infer that $a(5qr, n) = 0$.

Otherwise, there must exist two neighboring symbols $rf(\ell_1)$ and $rf(\ell_2)$ in the second row of the corresponding Table such that

$$rf(\ell_1) > n \ge rf(\ell_2).$$

If $0 \le \ell_2 \le 4$ (or $q \le \ell_2 \le q+4$), the value of $a(5qr, n)$ is given by computing the sum of values from $a_{f(\ell_2)}$ (or $-a_{f(\ell_2)}$) to the end of the relevant row. Next, we will explain the process of computing the value of $a(5qr, n)$ with the following example.

Example 4.1. Let $(p, q, r) = (5, 29, 701)$ and $n = 29589$. Note that $q \equiv -1 \pmod 5$ and $6r \equiv 1 \pmod{5q}$. By using congruence $rf(0) \equiv n \pmod{5q}$ and $0 \le f(0) \le 5q-1$, we deduce that $f(0) = 54$. This implies that

$$q + 24 \le f(0) \le 2q - 1 \text{ and } \overline{f(0)} = 4.$$

So this case corresponds to **the last row in Table 10**. Furthermore, it follows from (4.4)-(4.8) that $f(i) = 54 - 6i$ and $f(q+i) = 25 - 6i$, where $0 \le i \le 4$. This yields

$$rf(0) > rf(1) > n > rf(2) > rf(3) > rf(4) > rf(q) > \cdots > rf(q+4).$$

That is $\ell_1 = 1$ and $\ell_2 = 2$. Then the value of $a(5 \cdot 29 \cdot 701, 29589)$ is equal to the sum of values from $a_{f(2)}$ to the end of the last row in Table 10. Finally, on observing Table 10, we have

$$a(5 \cdot 29 \cdot 701, 29589) = 0 + (-1) + 0 + (-1) + 0 + 0 + 0 + 1 = -1.$$

It is easy to verify that the sum of values, from anywhere to the end of the row in all tables, is equal to -1, 0 or 1. Hence $a(5qr, n) \in \{-1, 0, 1\}$ for all $n \in [0, \phi(5qr)]$. Namely, $A(5qr) = 1$ in the case $q \ge 29$, $q \equiv -1 \pmod 5$ and $6r \equiv 1 \pmod{5q}$. This completes the proof of Proposition 3.1.

5. Proof of Proposition 3.6

This result can be proved by the same method as employed in Section 4. By using the PARI/GP system, we have $A(7 \cdot 29 \cdot 643) = 2$. Then, by Lemma 2.1, we get $A(7 \cdot 29 \cdot r) = 2$ for those primes r with $6r \equiv 6 \cdot 643 \equiv \pm 1 \pmod{7 \cdot 29}$.

Next we show that $A(7qr) = 1$, where $q > 29$, $q \equiv 1 \pmod 7$ and $6r \equiv 1 \pmod{7q}$. Note that Lemma 2.2 yields

$$a(7qr, n) = \sum_{i=0}^{6} a^*(7q, f(i)) + \sum_{i=q}^{q+6} \Big(- a^*(7q, f(i)) \Big), \tag{5.1}$$

where $f(i) \equiv r^{-1}(n - i) \pmod{7q}$, $0 \le f(i) \le 7q - 1$, and

$$a^*(7q, f(i)) = \begin{cases} a(7q, f(i)) & \text{if } rf(i) \le n; \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

For $7q + 1 = 7s + qt$, we have $s = \frac{6q+1}{7}$, $t = 1$. Thus in this case, we may rewrite the conclusion of Lemma 2.3 in the following form

$$a(7q, j) = \begin{cases} 1 & \text{if } j \equiv 0 \pmod 7 \text{ and } 0 \le j \le 6q - 6; \\ -1 & \text{if } j \equiv 1 \pmod 7 \text{ and } 1 \le j \le q - 7; \\ -1 & \text{if } j \equiv 2 \pmod 7 \text{ and } q + 1 \le j \le 2q - 7; \\ -1 & \text{if } j \equiv 3 \pmod 7 \text{ and } 2q + 1 \le j \le 3q - 7; \\ -1 & \text{if } j \equiv 4 \pmod 7 \text{ and } 3q + 1 \le j \le 4q - 7; \\ -1 & \text{if } j \equiv 5 \pmod 7 \text{ and } 4q + 1 \le j \le 5q - 7; \\ -1 & \text{if } j \equiv 6 \pmod 7 \text{ and } 5q + 1 \le j \le 6q - 7; \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

For any given $n \in [0, \phi(5qr)]$, the value of $f(i)$ is uniquely defined, since $rf(i) + i \equiv n \pmod{5q}$ and we have

$$f(q) \equiv f(0) + q \pmod{7q}, \tag{5.4}$$

$$f(1) \equiv f(0) - 6 \pmod{7q}, \qquad f(q+1) \equiv f(0) + q - 6 \pmod{7q}, \tag{5.5}$$

$$f(2) \equiv f(0) - 12 \pmod{7q}, \qquad f(q+2) \equiv f(0) + q - 12 \pmod{7q}, \tag{5.6}$$

$$f(3) \equiv f(0) - 18 \pmod{7q}, \qquad f(q+3) \equiv f(0) + q - 18 \pmod{7q}, \tag{5.7}$$

$$f(4) \equiv f(0) - 24 \pmod{7q}, \qquad f(q+4) \equiv f(0) + q - 24 \pmod{7q}, \tag{5.8}$$

$$f(5) \equiv f(0) - 30 \pmod{7q}, \qquad f(q+5) \equiv f(0) + q - 30 \pmod{7q}, \tag{5.9}$$

$$f(6) \equiv f(0) - 36 \pmod{7q}, \qquad f(q+6) \equiv f(0) + q - 36 \pmod{7q}. \tag{5.10}$$

In order to use (5.1) and (5.2), we need to determine for which i, $rf(i) \le n$. Now according to the value of $f(0)$, we give the following tables. The second row of each table is the inequality about $rf(i)$ for $i \in \{0, 1, 2, 3, 4, 5, 6, q, q+1, q+2, q+3, q+4, q+5, q+6\}$. In this section, for the reasons of space, we set

$$a_i := a(7q, f(i))$$

and let $\overline{f(0)}$ be the unique integer such that $0 \le \overline{f(0)} \le 6$ and $\overline{f(0)} \equiv f(0) \pmod{7}$. The values of a_i are obtained by using (5.3)–(5.10).

Table 1. $0 \le f(0) \le 5$

	$rf(1) > \cdots > rf(6) > rf(q) > \cdots > rf(q+6) > rf(0)$													
$f(0)$	a_1	a_2	a_3	a_4	a_5	a_6	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0
0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1
1	0	0	0	0	0	0	1	0	0	0	0	-1	1	-1
2	0	0	0	0	0	0	0	0	0	0	-1	1	0	0
3	0	0	0	0	0	0	0	0	0	-1	1	0	0	0
4	0	0	0	0	0	0	0	0	-1	1	0	0	0	0
5	0	0	0	0	0	0	0	-1	1	0	0	0	0	0

Table 2. $6 \le f(0) \le 11$

	$rf(2) > \cdots > rf(6) > rf(q) > \cdots > rf(q+6) > rf(0) > rf(1)$													
$f(0)$	a_2	a_3	a_4	a_5	a_6	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1
6	0	0	0	0	0	-1	0	0	0	0	0	0	0	1
7	0	0	0	0	0	0	1	0	0	0	0	-1	1	-1
8	0	0	0	0	0	1	0	0	0	0	-1	1	-1	0
9	0	0	0	0	0	0	0	0	0	-1	1	0	0	0
10	0	0	0	0	0	0	0	0	-1	1	0	0	0	0
11	0	0	0	0	0	0	0	-1	1	0	0	0	0	0

Table 3. $12 \le f(0) \le 17$

	$rf(3) \cdots > rf(6) > rf(q) > \cdots > rf(q+6) > rf(0) > rf(1) > rf(2)$													
$f(0)$	a_3	a_4	a_5	a_6	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2
12	0	0	0	0	0	-1	0	0	0	0	0	0	0	1
13	0	0	0	0	-1	0	1	0	0	0	0	0	1	-1
14	0	0	0	0	0	1	0	0	0	0	-1	1	-1	0
15	0	0	0	0	1	0	0	0	0	-1	1	-1	0	0
16	0	0	0	0	0	0	0	0	-1	1	0	0	0	0
17	0	0	0	0	0	0	0	-1	1	0	0	0	0	0

Table 4. $18 \le f(0) \le 23$

	$rf(4) > rf(5) > rf(6) > rf(q) > \cdots > rf(q+6) > rf(0) > rf(1) > rf(2) > rf(3)$													
$f(0)$	a_4	a_5	a_6	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2	a_3
18	0	0	0	0	0	-1	0	0	0	0	0	0	0	1
19	0	0	0	0	-1	0	1	0	0	0	0	0	1	-1
20	0	0	0	-1	0	1	0	0	0	0	0	1	-1	0
21	0	0	0	0	1	0	0	0	0	-1	1	-1	0	0
22	0	0	0	1	0	0	0	0	-1	1	-1	0	0	0
23	0	0	0	0	0	0	0	-1	1	0	0	0	0	0

Table 5. $24 \le f(0) \le 29$

	$rf(5) > rf(6) > rf(q) > \cdots > rf(q+6) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4)$													
$f(0)$	a_5	a_6	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2	a_3	a_4
24	0	0	0	0	0	-1	0	0	0	0	0	0	0	1
25	0	0	0	0	-1	0	1	0	0	0	0	0	1	-1
26	0	0	0	-1	0	1	0	0	0	0	0	1	-1	0
27	0	0	-1	0	1	0	0	0	0	0	1	-1	0	0
28	0	0	0	1	0	0	0	0	-1	1	-1	0	0	0
29	0	0	1	0	0	0	0	-1	1	-1	0	0	0	0

Table 6. $30 \le f(0) \le 35$

	$rf(6) > rf(q) > \cdots > rf(q+6) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(5)$													
$f(0)$	a_6	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2	a_3	a_4	a_5
30	1	0	0	0	0	-1	0	0	0	0	0	0	0	0
31	0	0	0	0	-1	0	1	0	0	0	0	0	1	-1
32	0	0	0	-1	0	1	0	0	0	0	0	1	-1	0
33	0	0	-1	0	1	0	0	0	0	0	1	-1	0	0
34	0	-1	0	1	0	0	0	0	0	1	-1	0	0	0
35	0	0	1	0	0	0	0	-1	1	-1	0	0	0	0

Table 7. $36 \le f(0) \le q-1$

	$rf(q) > \cdots > rf(q+6) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(5) > rf(6)$													
$\overline{f(0)}$	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2	a_3	a_4	a_5	a_6
0	0	1	0	0	0	0	-1	1	-1	0	0	0	0	0
1	1	0	0	0	0	-1	0	-1	0	0	0	0	0	1
2	0	0	0	0	-1	0	1	0	0	0	0	0	1	-1
3	0	0	0	-1	0	1	0	0	0	0	0	1	-1	0
4	0	0	-1	0	1	0	0	0	0	0	1	-1	0	0
5	0	-1	0	1	0	0	0	0	0	1	-1	0	0	0
6	-1	0	1	0	0	0	0	0	1	-1	0	0	0	0

Table 8. $q \le f(0) \le q+30$

	$rf(q) > rf(q+1) > \cdots > rf(q+6) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(5) > rf(6)$													
$f(0)$	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2	a_3	a_4	a_5	a_6
q	0	0	0	0	0	-1	0	0	0	0	0	0	0	1
$q+1$	1	0	0	0	-1	0	1	-1	0	0	0	0	1	-1
$q+2$	0	0	0	-1	0	1	0	0	0	0	0	1	-1	0
$q+3$	0	0	-1	0	1	0	0	0	0	0	1	-1	0	0
$q+4$	0	-1	0	1	0	0	0	0	0	1	-1	0	0	0
$q+5$	-1	0	1	0	0	0	0	0	1	-1	0	0	0	0
$q+6$	0	0	0	0	0	0	-1	1	0	0	0	0	0	0
$q+7$	0	1	0	0	0	-1	0	0	-1	0	0	0	0	1
$q+8$	1	0	0	0	-1	0	1	-1	0	0	0	0	1	-1
$q+9$	0	0	0	-1	0	1	0	0	0	0	0	1	-1	0
$q+10$	0	0	-1	0	1	0	0	0	0	0	1	-1	0	0
$q+11$	0	-1	0	1	0	0	0	0	0	1	-1	0	0	0
$q+12$	-1	0	0	0	0	0	0	0	1	0	0	0	0	0
$q+13$	0	0	1	0	0	0	-1	1	0	-1	0	0	0	0
$q+14$	0	1	0	0	0	-1	0	0	-1	0	0	0	0	1
$q+15$	1	0	0	0	-1	0	1	-1	0	0	0	0	1	-1
$q+16$	0	0	0	-1	0	1	0	0	0	0	0	1	-1	0
$q+17$	0	0	-1	0	1	0	0	0	0	0	1	-1	0	0
$q+18$	0	-1	0	0	0	0	0	0	0	1	0	0	0	0
$q+19$	-1	0	0	1	0	0	0	0	1	0	-1	0	0	0
$q+20$	0	0	1	0	0	0	-1	1	0	-1	0	0	0	0
$q+21$	0	1	0	0	0	-1	0	0	-1	0	0	0	0	1
$q+22$	1	0	0	0	-1	0	1	-1	0	0	0	0	1	-1
$q+23$	0	0	0	-1	0	1	0	0	0	0	0	1	-1	0
$q+24$	0	0	-1	0	0	0	0	0	0	0	1	0	0	0
$q+25$	0	-1	0	0	1	0	0	0	0	1	0	-1	0	0
$q+26$	-1	0	0	1	0	0	0	0	1	0	-1	0	0	0
$q+27$	0	0	1	0	0	0	-1	1	0	-1	0	0	0	0
$q+28$	0	1	0	0	0	-1	0	0	-1	0	0	0	0	1
$q+29$	1	0	0	0	-1	0	1	-1	0	0	0	0	1	-1
$q+30$	0	0	0	-1	0	0	0	0	0	0	0	1	0	0

Table 9. $q+31 \le f(0) \le 2q-1$

	$rf(q) > \cdots > rf(q+6) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(5) > rf(6)$													
$\overline{f(0)}$	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2	a_3	a_4	a_5	a_6
0	0	0	1	0	0	0	-1	1	0	-1	0	0	0	0
1	0	1	0	0	0	-1	0	0	-1	0	0	0	0	1
2	1	0	0	0	-1	0	0	-1	0	0	0	0	1	0
3	0	0	0	-1	0	0	1	0	0	0	0	1	0	-1
4	0	0	-1	0	0	1	0	0	0	0	1	0	-1	0
5	0	-1	0	1	0	0	0	0	0	1	-1	0	0	0
6	-1	0	1	0	0	0	0	0	1	0	-1	0	0	0

Table 10. $2q \leq f(0) \leq 2q + 30$

	$rf(q) > rf(q+1) > \cdots > rf(q+6) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(5) > rf(6)$													
$f(0)$	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2	a_3	a_4	a_5	a_6
$2q$	0	0	0	0	-1	0	0	0	0	0	0	0	1	0
$2q+1$	1	0	0	-1	0	0	1	-1	0	0	0	1	0	-1
$2q+2$	0	0	-1	0	0	1	0	0	0	0	1	0	-1	0
$2q+3$	0	-1	0	0	1	0	0	0	0	1	0	-1	0	0
$2q+4$	-1	0	0	1	0	0	0	0	1	0	-1	0	0	0
$2q+5$	0	0	1	0	0	0	-1	1	0	-1	0	0	0	0
$2q+6$	0	0	0	0	0	-1	0	0	0	0	0	0	0	1
$2q+7$	0	1	0	0	-1	0	0	0	-1	0	0	0	1	0
$2q+8$	1	0	0	-1	0	0	1	-1	0	0	0	1	0	-1
$2q+9$	0	0	-1	0	0	1	0	0	0	0	1	0	-1	0
$2q+10$	0	-1	0	0	1	0	0	0	0	1	0	-1	0	0
$2q+11$	-1	0	0	1	0	0	0	0	1	0	-1	0	0	0
$2q+12$	0	0	0	0	0	0	-1	1	0	0	0	0	0	0
$2q+13$	0	0	1	0	0	-1	0	0	0	-1	0	0	0	1
$2q+14$	0	1	0	0	-1	0	0	0	-1	0	0	0	1	0
$2q+15$	1	0	0	-1	0	0	1	-1	0	0	0	1	0	-1
$2q+16$	0	0	-1	0	0	1	0	0	0	0	1	0	-1	0
$2q+17$	0	-1	0	0	1	0	0	0	0	1	0	-1	0	0
$2q+18$	-1	0	0	0	0	0	0	0	1	0	0	0	0	0
$2q+19$	0	0	0	1	0	0	-1	1	0	0	-1	0	0	0
$2q+20$	0	0	1	0	0	-1	0	0	0	-1	0	0	0	1
$2q+21$	0	1	0	0	-1	0	0	0	-1	0	0	0	1	0
$2q+22$	1	0	0	-1	0	0	1	-1	0	0	0	1	0	-1
$2q+23$	0	0	-1	0	0	1	0	0	0	0	1	0	-1	0
$2q+24$	0	-1	0	0	0	0	0	0	0	1	0	0	0	0
$2q+25$	-1	0	0	0	1	0	0	0	1	0	0	-1	0	0
$2q+26$	0	0	0	1	0	0	-1	1	0	0	-1	0	0	0
$2q+27$	0	0	1	0	0	-1	0	0	0	-1	0	0	0	1
$2q+28$	0	1	0	0	-1	0	0	0	-1	0	0	0	1	0
$2q+29$	1	0	0	-1	0	0	1	-1	0	0	0	1	0	-1
$2q+30$	0	0	-1	0	0	0	0	0	0	0	1	0	0	0

Table 11. $2q + 31 \leq f(0) \leq 3q - 1$

	$rf(q) > \cdots > rf(q+6) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(5) > rf(6)$													
$\overline{f(0)}$	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2	a_3	a_4	a_5	a_6
0	0	0	0	1	0	0	-1	1	0	0	-1	0	0	0
1	0	0	1	0	0	-1	0	0	0	-1	0	0	0	1
2	0	1	0	0	-1	0	0	0	-1	0	0	0	1	0
3	1	0	0	-1	0	0	0	-1	0	0	0	1	0	0
4	0	0	-1	0	0	0	1	0	0	0	1	0	0	-1
5	0	-1	0	0	0	1	0	0	0	1	0	0	-1	0
6	-1	0	0	0	1	0	0	0	1	0	0	-1	0	0

Table 12. $3q \leq f(0) \leq 3q + 30$

	$rf(q) > rf(q+1) > \cdots > rf(q+6) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(5) > rf(6)$													
$f(0)$	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2	a_3	a_4	a_5	a_6
$3q$	0	0	0	-1	0	0	0	0	0	0	0	1	0	0
$3q+1$	1	0	-1	0	0	0	1	-1	0	0	1	0	0	-1
$3q+2$	0	-1	0	0	0	1	0	0	0	1	0	0	-1	0
$3q+3$	-1	0	0	0	1	0	0	0	1	0	0	-1	0	0
$3q+4$	0	0	0	1	0	0	-1	1	0	0	-1	0	0	0
$3q+5$	0	0	1	0	0	-1	0	0	0	-1	0	0	0	1
$3q+6$	0	0	0	0	-1	0	0	0	0	0	0	0	1	0
$3q+7$	0	1	0	-1	0	0	0	0	-1	0	0	1	0	0
$3q+8$	1	0	-1	0	0	0	1	-1	0	0	1	0	0	-1
$3q+9$	0	-1	0	0	0	1	0	0	0	1	0	0	-1	0
$3q+10$	-1	0	0	0	1	0	0	0	1	0	0	-1	0	0
$3q+11$	0	0	0	1	0	0	-1	1	0	0	-1	0	0	0
$3q+12$	0	0	0	0	0	-1	0	0	0	0	0	0	0	1
$3q+13$	0	0	1	0	-1	0	0	0	0	-1	0	0	1	0
$3q+14$	0	1	0	-1	0	0	0	0	-1	0	0	1	0	0
$3q+15$	1	0	-1	0	0	0	1	-1	0	0	1	0	0	-1
$3q+16$	0	-1	0	0	0	1	0	0	0	1	0	0	-1	0
$3q+17$	-1	0	0	0	1	0	0	0	1	0	0	-1	0	0
$3q+18$	0	0	0	0	0	0	-1	1	0	0	0	0	0	0
$3q+19$	0	0	0	1	0	-1	0	0	0	0	-1	0	0	1
$3q+20$	0	0	1	0	-1	0	0	0	0	-1	0	0	1	0
$3q+21$	0	1	0	-1	0	0	0	0	-1	0	0	1	0	0
$3q+22$	1	0	-1	0	0	0	1	-1	0	0	1	0	0	-1
$3q+23$	0	-1	0	0	0	1	0	0	0	1	0	0	-1	0
$3q+24$	-1	0	0	0	0	0	0	0	1	0	0	0	0	0
$3q+25$	0	0	0	0	1	0	-1	1	0	0	0	-1	0	0
$3q+26$	0	0	0	1	0	-1	0	0	0	0	-1	0	0	1
$3q+27$	0	0	1	0	-1	0	0	0	0	-1	0	0	1	0
$3q+28$	0	1	0	-1	0	0	0	0	-1	0	0	1	0	0
$3q+29$	1	0	-1	0	0	0	1	-1	0	0	1	0	0	-1
$3q+30$	0	-1	0	0	0	0	0	0	0	1	0	0	0	0

Table 13. $3q + 31 \leq f(0) \leq 4q - 1$

	$rf(q) > \cdots > rf(q+6) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(5) > rf(6)$													
$\overline{f(0)}$	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2	a_3	a_4	a_5	a_6
0	0	0	0	0	1	0	-1	1	0	0	0	-1	0	0
1	0	0	0	1	0	-1	0	0	0	0	-1	0	0	1
2	0	0	1	0	-1	0	0	0	0	-1	0	0	1	0
3	0	1	0	-1	0	0	0	0	-1	0	0	1	0	0
4	1	0	-1	0	0	0	0	-1	0	0	1	0	0	0
5	0	-1	0	0	0	0	1	0	0	1	0	0	0	-1
6	-1	0	0	0	0	1	0	0	1	0	0	0	-1	0

Table 14. $4q \leq f(0) \leq 4q + 30$

	$rf(q) > rf(q+1) > \cdots > rf(q+6) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(5) > rf(6)$													
$f(0)$	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2	a_3	a_4	a_5	a_6
$4q$	0	0	-1	0	0	0	0	0	0	0	1	0	0	0
$4q+1$	1	-1	0	0	0	0	1	-1	0	1	0	0	0	-1
$4q+2$	-1	0	0	0	0	1	0	0	1	0	0	0	-1	0
$4q+3$	0	0	0	0	1	0	-1	1	0	0	0	-1	0	0
$4q+4$	0	0	0	1	0	-1	0	0	0	0	-1	0	0	1
$4q+5$	0	0	1	0	-1	0	0	0	0	-1	0	0	1	0
$4q+6$	0	0	0	-1	0	0	0	0	0	0	0	1	0	0
$4q+7$	0	1	-1	0	0	0	0	0	-1	0	1	0	0	0
$4q+8$	1	-1	0	0	0	0	1	-1	0	1	0	0	0	-1
$4q+9$	-1	0	0	0	0	1	0	0	1	0	0	0	-1	0
$4q+10$	0	0	0	0	1	0	-1	1	0	0	0	-1	0	0
$4q+11$	0	0	0	1	0	-1	0	0	0	0	-1	0	0	1
$4q+12$	0	0	0	0	-1	0	0	0	0	0	0	0	1	0
$4q+13$	0	0	1	-1	0	0	0	0	0	-1	0	1	0	0
$4q+14$	0	1	-1	0	0	0	0	0	-1	0	1	0	0	0
$4q+15$	1	-1	0	0	0	0	1	-1	0	1	0	0	0	-1
$4q+16$	-1	0	0	0	0	1	0	0	1	0	0	0	-1	0
$4q+17$	0	0	0	0	1	0	-1	1	0	0	0	-1	0	0
$4q+18$	0	0	0	0	0	-1	0	0	0	0	0	0	0	1
$4q+19$	0	0	0	1	-1	0	0	0	0	0	-1	0	1	0
$4q+20$	0	0	1	-1	0	0	0	0	0	-1	0	1	0	0
$4q+21$	0	1	-1	0	0	0	0	0	-1	0	1	0	0	0
$4q+22$	1	-1	0	0	0	0	1	-1	0	1	0	0	0	-1
$4q+23$	-1	0	0	0	0	1	0	0	1	0	0	0	-1	0
$4q+24$	0	0	0	0	0	0	-1	1	0	0	0	0	0	0
$4q+25$	0	0	0	0	1	-1	0	0	0	0	0	-1	0	1
$4q+26$	0	0	0	1	-1	0	0	0	0	0	-1	0	1	0
$4q+27$	0	0	1	-1	0	0	0	0	0	-1	0	1	0	0
$4q+28$	0	1	-1	0	0	0	0	0	-1	0	1	0	0	0
$4q+29$	1	-1	0	0	0	0	1	-1	0	1	0	0	0	-1
$4q+30$	-1	0	0	0	0	0	0	0	1	0	0	0	0	0

Table 15. $4q + 31 \leq f(0) \leq 5q - 1$

	$rf(q) > \cdots > rf(q+6) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(5) > rf(6)$													
$\overline{f(0)}$	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2	a_3	a_4	a_5	a_6
0	0	0	0	0	0	1	-1	1	0	0	0	0	-1	0
1	0	0	0	0	1	-1	0	0	0	0	0	-1	0	1
2	0	0	0	1	-1	0	0	0	0	0	-1	0	1	0
3	0	0	1	-1	0	0	0	0	0	-1	0	1	0	0
4	0	1	-1	0	0	0	0	0	-1	0	1	0	0	0
5	0	-1	0	1	0	0	0	-1	0	1	0	0	0	0
6	-1	0	1	0	0	0	0	0	1	0	0	0	0	-1

Table 16. $5q \le f(0) \le 5q + 30$														
	$rf(q) > rf(q+1) > \cdots > rf(q+6) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(5) > rf(6)$													
$f(0)$	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2	a_3	a_4	a_5	a_6
$5q$	0	-1	0	0	0	0	0	0	0	1	0	0	0	0
$5q+1$	0	0	0	0	0	0	1	-1	1	0	0	0	0	-1
$5q+2$	0	0	0	0	0	1	-1	1	0	0	0	0	-1	0
$5q+3$	0	0	0	0	1	-1	0	0	0	0	0	-1	0	1
$5q+4$	0	0	0	1	-1	0	0	0	0	0	-1	0	1	0
$5q+5$	0	0	1	-1	0	0	0	0	0	-1	0	1	0	0
$5q+6$	0	0	-1	0	0	0	0	0	0	0	1	0	0	0
$5q+7$	0	0	0	0	0	0	0	0	-1	1	0	0	0	0
$5q+8$	0	0	0	0	0	0	1	-1	1	0	0	0	0	-1
$5q+9$	0	0	0	0	0	1	-1	1	0	0	0	0	-1	0
$5q+10$	0	0	0	0	1	-1	0	0	0	0	0	-1	0	1
$5q+11$	0	0	0	1	-1	0	0	0	0	0	-1	0	1	0
$5q+12$	0	0	0	-1	0	0	0	0	0	0	0	1	0	0
$5q+13$	0	0	0	0	0	0	0	0	0	-1	1	0	0	0
$5q+14$	0	0	0	0	0	0	0	0	-1	1	0	0	0	0
$5q+15$	0	0	0	0	0	0	1	-1	1	0	0	0	0	-1
$5q+16$	0	0	0	0	0	1	-1	1	0	0	0	0	-1	0
$5q+17$	0	0	0	0	1	-1	0	0	0	0	0	-1	0	1
$5q+18$	0	0	0	0	-1	0	0	0	0	0	0	0	1	0
$5q+19$	0	0	0	-1	0	0	0	0	0	0	0	1	0	0
$5q+20$	0	0	0	0	0	0	0	0	0	-1	1	0	0	0
$5q+21$	0	0	0	0	0	0	0	0	-1	1	0	0	0	0
$5q+22$	0	0	0	0	0	0	1	-1	1	0	0	0	0	-1
$5q+23$	0	0	0	0	0	1	-1	1	0	0	0	0	-1	0
$5q+24$	0	0	0	0	0	-1	0	0	0	0	0	0	0	1
$5q+25$	0	0	0	0	0	0	0	0	0	0	0	-1	1	0
$5q+26$	0	0	0	0	0	0	0	0	0	0	-1	1	0	0
$5q+27$	0	0	0	0	0	0	0	0	0	-1	1	0	0	0
$5q+28$	0	0	0	0	0	0	0	0	-1	1	0	0	0	0
$5q+29$	0	0	0	0	0	0	1	-1	1	0	0	0	0	-1
$5q+30$	0	0	0	0	0	0	-1	1	0	0	0	0	0	0

Table 17. $5q + 31 \le f(0) \le 6q - 1$														
	$rf(q) > \cdots > rf(q+6) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > rf(5) > rf(6)$													
$\overline{f(0)}$	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2	a_3	a_4	a_5	a_6
0	0	0	0	0	0	0	0	1	0	0	0	0	0	-1
1	0	0	0	0	0	0	0	0	0	0	0	0	-1	1
2	0	0	0	0	0	0	0	0	0	0	0	-1	1	0
3	0	0	0	0	0	0	0	0	0	0	-1	1	0	0
4	0	0	0	0	0	0	0	0	0	- 1	1	0	0	0
5	0	0	0	0	0	0	0	0	-1	1	0	0	0	0
6	0	0	0	0	0	0	0	-1	1	0	0	0	0	0

Table 18. $6q \le f(0) \le 6q + 5$														
	$rf(q+1) > \cdots > rf(q+6) > rf(0) > \cdots > rf(6) > rf(q)$													
$f(0)$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	$-a_q$
$6q$	0	0	0	0	0	0	0	1	0	0	0	0	0	-1
$6q+1$	0	0	0	0	0	0	0	0	0	0	0	0	-1	1
$6q+2$	0	0	0	0	0	0	0	0	0	0	0	-1	1	0
$6q+3$	0	0	0	0	0	0	0	0	0	0	-1	1	0	0
$6q+4$	0	0	0	0	0	0	0	0	0	-1	1	0	0	0
$6q+5$	0	0	0	0	0	0	0	0	-1	1	0	0	0	0

Table 19. $6q + 6 \le f(0) \le 6q + 11$														
	$rf(q+2) > \cdots > rf(q+6) > rf(0) > \cdots > rf(6) > rf(q) > rf(q+1)$													
$f(0)$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	$-a_q$	$-a_{q+1}$
$6q+6$	0	0	0	0	0	0	0	1	0	0	0	0	0	-1
$6q+7$	0	0	0	0	0	0	0	0	0	0	0	0	-1	1
$6q+8$	0	0	0	0	0	0	0	0	0	0	0	-1	1	0
$6q+9$	0	0	0	0	0	0	0	0	0	0	-1	1	0	0
$6q+10$	0	0	0	0	0	0	0	0	0	-1	1	0	0	0
$6q+11$	0	0	0	0	0	0	0	0	-1	1	0	0	0	0

Table 20. $6q + 12 \le f(0) \le 6q + 17$														
	$rf(q+3) > \cdots > rf(q+6) > rf(0) > \cdots > rf(6) > rf(q) > rf(q+1) > rf(q+2)$													
$f(0)$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	$-a_q$	$-a_{q+1}$	$-a_{q+2}$
$6q+12$	0	0	0	0	0	0	0	1	0	0	0	0	0	-1
$6q+13$	0	0	0	0	0	0	0	0	0	0	0	0	-1	1
$6q+14$	0	0	0	0	0	0	0	0	0	0	0	-1	1	0
$6q+15$	0	0	0	0	0	0	0	0	0	0	-1	1	0	0
$6q+16$	0	0	0	0	0	0	0	0	0	-1	1	0	0	0
$6q+17$	0	0	0	0	0	0	0	0	-1	1	0	0	0	0

Table 21. $6q + 18 \le f(0) \le 6q + 23$														
	$rf(q+4) > rf(q+5) > rf(q+6) > rf(0) > \cdots > rf(6) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3)$													
$f(0)$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$
$6q+18$	0	0	0	0	0	0	0	1	0	0	0	0	0	-1
$6q+19$	0	0	0	0	0	0	0	0	0	0	0	0	-1	1
$6q+20$	0	0	0	0	0	0	0	0	0	0	0	-1	1	0
$6q+21$	0	0	0	0	0	0	0	0	0	0	-1	1	0	0
$6q+22$	0	0	0	0	0	0	0	0	0	-1	1	0	0	0
$6q+23$	0	0	0	0	0	0	0	0	-1	1	0	0	0	0

Table 22. $6q+24 \le f(0) \le 6q+29$

	$rf(q+5) > rf(q+6) > rf(0) > \cdots > rf(6) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4)$													
$f(0)$	$-a_{q+5}$	$-a_{q+6}$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$
$6q+24$	0	0	0	0	0	0	0	1	0	0	0	0	0	-1
$6q+25$	0	0	0	0	0	0	0	0	0	0	0	0	-1	1
$6q+26$	0	0	0	0	0	0	0	0	0	0	0	-1	1	0
$6q+27$	0	0	0	0	0	0	0	0	0	0	-1	1	0	0
$6q+28$	0	0	0	0	0	0	0	0	0	-1	1	0	0	0
$6q+29$	0	0	0	0	0	0	0	0	-1	1	0	0	0	0

Table 23. $6q+30 \le f(0) \le 6q+35$

	$rf(q+6) > rf(0) > \cdots > rf(6) > rf(q) > rf(q+1) > rf(q+2) > rf(q+3) > rf(q+4) > rf(q+5)$													
$f(0)$	$-a_{q+6}$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$
$6q+30$	0	0	0	0	0	0	0	1	0	0	0	0	0	-1
$6q+31$	0	0	0	0	0	0	0	0	0	0	0	0	-1	1
$6q+32$	0	0	0	0	0	0	0	0	0	0	0	-1	1	0
$6q+33$	0	0	0	0	0	0	0	0	0	0	-1	1	0	0
$6q+34$	0	0	0	0	0	0	0	0	0	-1	1	0	0	0
$6q+35$	0	0	0	0	0	0	0	0	-1	1	0	0	0	0

Table 24. $6q+36 \le f(0) \le 7q-1$

	$rf(0) > \cdots > rf(6) > rf(q) > \cdots > rf(q+6)$													
$\overline{f(0)}$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	$-a_q$	$-a_{q+1}$	$-a_{q+2}$	$-a_{q+3}$	$-a_{q+4}$	$-a_{q+5}$	$-a_{q+6}$
0	0	0	0	0	0	0	0	1	0	0	0	0	0	-1
1	0	0	0	0	0	0	0	0	0	0	0	0	-1	1
2	0	0	0	0	0	0	0	0	0	0	0	-1	1	0
3	0	0	0	0	0	0	0	0	0	0	-1	1	0	0
4	0	0	0	0	0	0	0	0	0	-1	1	0	0	0
5	0	0	0	0	0	0	0	0	-1	1	0	0	0	0
6	0	0	0	0	0	0	0	-1	1	0	0	0	0	0

It is a routine matter to check that the sum of values, from anywhere to the end of the row in all tables, is equal to -1, 0 or 1. That is to say, the data reveals that the sums in (5.1) are always in the set $\{-1, 0, 1\}$. So $A(7qr) = 1$ in the case $q \ge 43$, $q \equiv 1 \pmod 5$ and $6r \equiv 1 \pmod{7q}$. This completes the proof of Proposition 3.6.

6. Proofs of Propositions 3.2-3.5 and 3.7

Proof of Proposition 3.2.

(1) By using the PARI/GP system, we have $A(7 \cdot 13 \cdot 167) = 2$. Then, by Lemma 2.1, we get $A(7 \cdot 13 \cdot r) = 2$ for those primes r with $6r \equiv 6 \cdot 167 \equiv \pm 1 \pmod{7 \cdot 13}$.

(2) Let $n = 4qr + 5$. In order to use Lemma 2.2 to prove $a(7qr, n) = 2$, we need to determine for which i will $rf(i) > n$. Applying n to congruence

$$rf(i) + i \equiv n \pmod{7q}$$

and noting $0 \le f(i) \le 7q - 1$, we have

$$f(i) = 4q + 30 - 6i \text{ and } f(q+i) = 5q + 30 - 6i,$$

where $0 \le i \le 6$. So

$$rf(q) > \cdots > rf(q{+}6) > rf(0) > rf(1) > rf(2) > rf(3) > rf(4) > n > rf(5) > rf(6).$$

Then

$$a^*(5q, f(i)) = \begin{cases} a(5q, f(i)) & \text{if } i \in \{5, 6\}; \\ 0 & \text{if } 0 \le i \le 4 \text{ or } q \le i \le q+6. \end{cases}$$

Note that $f(5) = 4 \cdot q$ and $f(6) = \frac{q-6}{7} \cdot 7 + 3 \cdot q$. It follows from Lemma 2.4 that $a(7q, f(5)) = a(7q, f(6)) = 1$. Hence, by Lemma 2.2,

$$a(7qr, n) = a(7q, f(5)) + a(7q, f(6)) = 2.$$

Proof of Proposition 3.3.

Let $n = 5qr + 33r + 1$. By substituting n into congruence (2.1) and using $0 \le f(i) \le 7q - 1$, we have $f(i) = 5q + 39 - 6i$ and $f(q + i) = 10q + 39 - 6i$, where $0 \le i \le 10$. This implies that $rf(i) > n$ when $i = 0$ or $q \le i \le q + 10$, and $n \ge rf(i)$ when $1 \le i \le 10$. Then it follows from Lemma 2.1 that

$$a(11qr, n) = \sum_{i=1}^{10} a(11q, f(i)).$$

On noting that $f(1) = 3 \cdot 11 + 5 \cdot q$ and $f(10) = \frac{q-21}{11} \cdot 11 + 4 \cdot q$, we obtain $a(11q, f(1)) = a(11q, f(10)) = 1$ from Lemma 2.4. Then

$$a(11qr, n) = 2 + \sum_{i=2}^{9} a(11q, f(i)). \tag{6.1}$$

Let $2 \le i \le 9$. It is clear that the coefficient $a(11q, f(i))$ only takes on one of three values: -1, 1 or 0.

If $a(11q, f(i)) = -1$, then, by Lemma 2.4, we infer that

$$f(i) = 5q + 39 - 6i \equiv 1 \pmod{11}.$$

This yields $i = 0$, contradicting the range $2 \le i \le 9$.

If $a(11q, f(i)) = 1$, then, by Lemma 2.4, we infer that there exist $0 \le u \le \frac{q+1}{11}$ and $0 \le v \le 9$ such that

$$f(i) = 5q + 39 - 6i = 11u + vq.$$

Taking this equality modulo q gives

$$11u + 6i - 39 \equiv 0 \pmod{q}.$$

Since $0 \le u \le \frac{q+1}{11}$ and $2 \le i \le 9$, we get

$$-q < -39 \le 11u + 6i - 39 \le q + 5 < 2q.$$

So $11u + 6i - 39 = 0$ or q. If $11u + 6i - 39 = 0$, then we deduce that $u = 3$ and $i = 1$. If $11u + 6i - 39 = q$, then we deduce that $i \equiv 10 \pmod{11}$ and then $i = 10$. Both of these two cases contradict the range $2 \le i \le 9$.

Hence we obtain that $a(11q, f(i)) = 0$ for $2 \le i \le 9$. Combining this with (6.1) gives $a(11qr, n) = 2$.

Proof of Proposition 3.4.

(1) Let $n = 4qr + q + \frac{5p-5}{6}$. By using $rf(i) + i \equiv n \pmod{pq}$ and $0 \le f(i) \le pq - 1$, we have

$$f(i) = 10q + 5p - 5 - 6i,$$

where $i \in [0, p-1] \cup [q, q+p-1]$. Then $rf(i) > n$ when $i \in [0, p-1] \cup [q, q + \frac{5p-11}{6}]$ and $n > rf(i)$ when $i \in [q + \frac{5p-5}{6}, q+p-1]$. Then it follows from Lemma 2.1 that

$$a(pqr, n) = -\sum_{i=\frac{5p-5}{6}}^{p-1} a(pq, f(q+i)). \tag{6.2}$$

On invoking that $f(q + \frac{5p-5}{6}) = 4 \cdot q$ and $f(q+p-1) = \frac{q-p+1}{p} \cdot p + 3 \cdot q$, we infer that $a(pq, f(q + \frac{5p-5}{6})) = a(pq, f(q+p-1)) = 1$. Thus equation (6.2) becomes

$$a(pqr, n) = -2 - \sum_{i=\frac{5p+1}{6}}^{p-2} a(pq, f(q+i)).$$

Recall that the coefficients of binary cyclotomic polynomials only take on one of three values: -1, 1 or 0. If $a(pq, f(q+i)) = -1$ for $\frac{5p+1}{6} \le i \le p-2$, then, by Lemma 2.4, we have

$$f(q+i) = 4q + 5p - 5 - 6i \equiv 1 \pmod{p}$$

This yields

$$6i + 10 \equiv 0 \pmod{p}. \tag{6.3}$$

But it follows from $\frac{5p+1}{6} \le i \le p-2$ that $5p+11 \le 6i+10 \le 6p-2$, a contradiction to (6.3).

If $a(pq, f(q+i)) = 1$ for $\frac{5p+1}{6} \le i \le p-2$, then there must exist $0 \le u \le \frac{q+1}{p} - 1$ and $0 \le v \le p-2$ such that

$$f(q+i) = 4q + 5p - 5 - 6i = up + vq.$$

Taking this equality modulo q gives

$$up + 5 + 6i - 5p \equiv 0 \pmod{q}. \tag{6.4}$$

By using $0 \le u \le \frac{q+1}{p} - 1$ and $\frac{5p+1}{6} \le i \le p-2$, we obtain

$$6 \le up + 5 + 6i - 5p \le q - 6.$$

This is a contradiction to (6.4) and yields $a(pq, f(q+i)) = 0$ for $\frac{5p+1}{6} \le i \le p-2$. So $a(pqr, n) = -2$.

(2) Let $n = 3pr + 5qr + q + \frac{p-5}{6}$. Applying this to congruence (2.1), we obtain

$$f(i) = 4p + 11q - 5 - 6i. \tag{6.5}$$

Then one readily verifies that $rf(i) > n$ when $i \in [0, p-1] \cup [q, q+\frac{p-11}{6}]$ and $n > rf(i)$ when $i \in [q+\frac{p-5}{6}, q+p-1]$. On noting that $f(q+\frac{p-5}{6}) = 3\cdot p + 5\cdot q$ and $f(q+p-1) = \frac{q-2p+1}{p}\cdot p + 4\cdot q$, we infer from Lemmas 2.4 and 2.1 that

$$a(pqr, n) = -2 - \sum_{i=\frac{p+1}{6}}^{p-2} a(pq, f(q+i)). \tag{6.6}$$

Let $\frac{p+1}{6} \le i \le p-2$. It follows from (6.5) and $p \equiv 5 \pmod 6$ that $f(q+i) \not\equiv 1 \pmod p$. Then, by Lemma 2.4, we have $a(pq, f(q+i)) \ne -1$.

Next we show that $a(pq, f(q+i)) \ne 1$ for $\frac{p+1}{6} \le i \le p-2$. If not, then there must exist $0 \le u \le \frac{q+1}{p} - 1$ and $0 \le v \le p-2$ such that

$$f(q+i) = 4p + 5q - 5 - 6i = up + vq.$$

Taking this equality modulo q gives

$$up + 5 + 6i - 4p \equiv 0 \pmod q. \tag{6.7}$$

By using $0 \le u \le \frac{q+1}{p} - 1$ and $\frac{p+1}{6} \le i \le p-2$, we obtain

$$-3p + 6 \le up + 5 + 6i - 5p \le q + p - 6. \tag{6.8}$$

Note that the condition $p \equiv 5 \pmod 6$ implies $q \ne 2p-1$. Combining this with (6.7) and (6.8) yields $up + 5 + 6i - 5p = 0$ or q. If $up + 5 + 6i - 5p = 0$, then we get $6i + 5 \equiv 0 \pmod p$. It follows from $\frac{p+1}{6} \le i \le p-2$ that $6i + 5 = 2p$, $3p$, $4p$ or $5p$. It is easy to see that all these four cases are impossible, since $p \equiv 5 \pmod 6$. Similarly, we can show that $up + 5 + 6i - 5p \ne q$.

It follows from Lemma 2.4 that $a(pq, f(q+i)) = 0$ for $\frac{p+1}{6} \le i \le p-2$ and then $a(pqr, n) = -2$.

Proof of Proposition 3.5.

Let $n = qr + q + r + 3$. Applying n to congruence $rf(i) + i \equiv n \pmod{5q}$, we have $f(0) = 2q+19$, $f(1) = 2q+13$, $f(2) = 2q+7$, $f(3) = 2q+1$, $f(4) = 2q-5$; $f(q) = q+19$, $f(q+1) = q+13$, $f(q+2) = q+7$, $f(q+3) = q+1$ and $f(q+4) = q-5$. So

$$rf(0) > \cdots > rf(4) > rf(q) > rf(q+1) > rf(q+2) > n > rf(q+3) > rf(q+4).$$

Then

$$a^*(5q, f(i)) = \begin{cases} a(5q, f(i)) & \text{if } i \in \{q+3, q+4\}; \\ 0 & \text{if } i \in \{0,1,2,3,4,q,q+1,q+2\}. \end{cases}$$

Note that $f(q+3) = q\cdot 1 + 1$ and $f(q+4) = \frac{q-6}{5}\cdot 5 + 1$. It follows from Lemma 2.5 that $a(5q, f(q+3)) = a(5q, f(q+4)) = -1$. Hence, by Lemma 2.2,

$$a(5qr, n) = -a(5q, f(q+3)) - a(5q, f(q+4)) = 2.$$

Proof of Proposition 3.7.

The proof is similar to the proofs of Propositions 3.2-3.5. We omit the straightforward details.

Acknowledgements. We would like to thank the referee for helpful suggestions.

References

[1] G. Bachman, *Flat cyclotomic polynomials of order three*, Bull. London Math. Soc., **38** (2006), 53–60.

[2] M. Beiter, *Coefficients of the cyclotomic polynomial $F_{3qr}(x)$*, Fibonacci Quart, **16** (4)(1978), 302–306.

[3] S. Elder, *Flat Cyclotomic Polynomials: A New Approach*, arXiv:1207.5811v1, 2012.

[4] T. Flanagan, *On the coefficients of ternary cyclotomic polynomials*, MS Thesis, University of Nevada Las Vegas, 2006.

[5] C.G. Ji, *A special family of cyclotomic polynomials of order three*, Science China Math, **53** (2010), 2269–2274.

[6] N. Kaplan, *Flat cyclotomic polynomials of order three*, J. Number Theory, **127** (2007), 118–126.

[7] T.Y. Lam, K.H. Leung, *On the cyclotomic polynomial $\Phi_{pq}(X)$*, Amer. Math. Monthly, **103** (1996), 562–564.

[8] H.W. Lenstra, *Vanishing sums of roots of unity*, Proceedings, Bicentennial Congress Wiskundig Genootschap (Vrije Univ., Amsterdam, 1978), Part II,, (1979), 249–268.

[9] R. Thangadurai, *On the coefficients of cyclotomic polynomials*, in: Cyclotomic Fields and Related Topics, Pune, 1999, Bhaskaracharya Pratishthana, Pune, (2000), 311–322.

[10] B. Zhang, Y. Zhou, *On a class of ternary cyclotomic polynomials*, Bull. Korean Math. Soc., **52** (6) (2015), 1911–1924.

[11] B. Zhang, *Remarks on the flatness of ternary cyclotomic polynomials*, Int. J Number Theory, **13** (2017), 529–547.

[12] B. Zhang, *The upper bound of a class of ternary cyclotomic polynomials*, Bull. Math. Soc. Sci. Math. Roumanie, **60** (1) (2017), 25–32.

[13] B. Zhang, *The height of a class of ternary cyclotomic polynomials*, Bull. Korean Math. Soc., **54** (1) (2017), 43–50.

[14] J. Zhao, X.K. Zhang, *Coefficients of ternary cyclotomic polynomials*, J. Number Theory, **130** (2010), 2223–2237.

Received submission date; revised revision date