On an open problem about $\pi$-quasi-$\mathfrak{F}$-groups

CHI ZHANG – LI ZHANG*– JIANHONG HUANG**

Abstract – Let $\mathfrak{F}$ be a class of finite groups, $p$ a prime and $\pi$ a set of some primes. A finite group $G$ is called a $p$-quasi-$\mathfrak{F}$-group (respectively, by $\pi$-quasi-$\mathfrak{F}$-group) provided that for every $\mathfrak{F}$-eccentric $G$-chief factor $H/K$ of order divisible by $p$ (respectively, by at least one prime in $\pi$), the automorphisms of $H/K$ induced by all elements of $G$ are inner. In this paper, we obtain the characterizations of $p$-quasi-$\mathfrak{F}$-groups and $\pi$-quasi-$\mathfrak{F}$-groups, which give a positive answer to an open problem in the book [3].

Mathematics Subject Classification (2010). 20D10, 20D15, 20D20

Keywords. Finite groups; $p$-quasinilpotent group; Fitting formation; $p$-quasi-$\mathfrak{F}$-group, $\pi$-quasi-$\mathfrak{F}$-group.

1. Introduction

Throughout this paper, all groups are finite. $G$ always denotes a group, $p$ is a prime and $\pi(G)$ is the set of all prime divisors of $|G|$. As usual, we use $\mathfrak{S}_p$ and $\mathfrak{N}$ to denote the class of all $p$-groups and the class of all nilpotent groups, respectively. All unexplained notation and terminology are standard, as in [2, 3, 9].

Recall that a class $\mathfrak{F}$ of groups is called a formation if $\mathfrak{F}$ is closed under taking homomorphic images and subdirect products. A formation $\mathfrak{F}$ is said to be:

*Research was supported by the NNSF of China (11371335) and Wu Wen-Tsun Key Laboratory of Mathematics of Chinese Academy of Sciences.

**Research was supported by the NNSF of China (11401264).

Chi Zhang, Department of Mathematics, University of Science and Technology of China, Hefei, 230026, P.R. China
E-mail: zcqxj32@mail.ustc.edu.cn

Li Zhang, Department of Mathematics, University of Science and Technology of China, Hefei, 230026, P.R. China
E-mail: zhang120@mail.ustc.edu.cn

Jianhong Huang, School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, 221116, P.R. China
E-mail: jhh320@126.com
(1) saturated (solubly saturated) if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$ (respectively, $G/\Phi(N) \in \mathfrak{F}$ for some soluble normal subgroup $N$ of $G$); (2) hereditary (normally hereditary) if $H \in \mathfrak{F}$ whenever $H \leq G \in \mathfrak{F}$ (respectively, $H \leq G \in \mathfrak{F}$). It is well known that a $G$-chief factor $H/K$ is called $\mathfrak{F}$-central in $G$ provided that $H/K \cong G/C_G(H/K) \in \mathfrak{F}$. Otherwise, it is called $\mathfrak{F}$-eccentric. Following [8], a normal subgroup $N$ of $G$ is said to be $p\mathfrak{F}$-hypercentral (respectively, $\pi\mathfrak{F}$-hypercentral) in $G$ if either $N = 1$ or $N \neq 1$ and every $G$-chief factor below $N$ of order divisible by $p$ (respectively, by at least one prime in $\pi$) is $\mathfrak{F}$-central in $G$. The symbol $Z_{p\mathfrak{F}}(G)$ denotes the $p\mathfrak{F}$-hypercentre of $G$, that is, the product of all normal $p\mathfrak{F}$-hypercentral subgroups of $G$, and the symbol $Z_{\pi\mathfrak{F}}(G)$ denotes the $\pi\mathfrak{F}$-hypercentre of $G$, that is, the product of all normal $\pi\mathfrak{F}$-hypercentral subgroups of $G$. Also, the $\mathfrak{F}$-hypercentre of $G$ denoted by $Z_\mathfrak{F}(G)$ (see [2, p.389]), is the product of all normal subgroups $N$ of $G$, where $N$ is $p\mathfrak{F}$-hypercentral in $G$ for every prime $p$. In particular, if $\mathfrak{F} = \mathfrak{N}$, then $Z_\mathfrak{N}(G)$ is the hypercentre of $G$ and is often denoted by $Z_\infty(G)$.

A group $G$ is said to be quasinilpotent [9, Chap. X, Definition 13.2] if for every $G$-chief factor $H/K$ and $x \in G$, $x$ induces an inner automorphism on $H/K$. A group $G$ is said to be $p$-quasinilpotent [10] if for every $G$-chief factor $R/L$ of order divisible by $p$ and $x \in G$, $x$ induces an inner automorphism on $R/L$.

As a important generalization of quasinilpotent groups and $p$-quasinilpotent groups, Guo and Skiba [5, 6] introduced the following notion:

**Definition 1.1.** [5, 6] (see also [3, Chap. 1, Definition 3.2]) Let $\mathfrak{F}$ be a class of groups, $G$ a group and $p$ a prime. Assume that $\emptyset \neq \pi \subseteq \pi(G)$. $G$ is called a $p$-quasi-$\mathfrak{F}$-group (respectively, $\pi$-quasi-$\mathfrak{F}$-group) if for every $\mathfrak{F}$-eccentric $G$-chief factor $H/K$ of order divisible by $p$ (respectively, at least one prime in $\pi$), the automorphisms of $H/K$ induced by all elements of $G$ are inner. In particular, if $\pi = \mathbb{P}$ is the set of all primes, then a $\pi$-quasi-$\mathfrak{F}$-group is called a quasi-$\mathfrak{F}$-group.

Following [5, 3], we use $\mathfrak{F}^*$, $\mathfrak{F}^*_p$ and $\mathfrak{F}^*_\pi$ to denote the class of all quasi-$\mathfrak{F}$-groups, $p$-quasi-$\mathfrak{F}$-groups and $\pi$-quasi-$\mathfrak{F}$-groups, respectively. Clearly, $\mathfrak{F}^* \subseteq \mathfrak{F}^*_\pi \subseteq \mathfrak{F}^*_p$. For the details, one can refer to [3, Chap. 1, §1.3].

In [5], Guo and Skiba have given the general theory of the quasi-$\mathfrak{F}$-groups and obtained some characterizations of quasisoluble groups and quasisupersoluble groups. In particular, they proved the following theorem.

**Theorem 1.2.** [5, Theorem B] Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{N}$. Then $G$ is a quasi-$\mathfrak{F}$-group if and only if for every $\mathfrak{F}$-eccentric $G$-chief factor $H/K$ between $\Phi(F(G))$ and $F^*(G)$, the automorphisms of $H/K$ induced by all elements of $G$ are inner.

Here, $F^*(G)$ is the generalized Fitting subgroup of $G$ (see [9, Chap. X, Definition 13.9]), that is, the set of all elements of $G$ which induce inner automorphisms on every $G$-chief factor. Clearly, $F^*(G)$ is a characteristic subgroup of $G$.

In connection with this, the author in the book [3] proposed the following open problem:
On an open problem about $\pi$-quasi-$\mathcal{F}$-groups

Problem. [3, Chap. I, Problem 6.1] Could we generalize the result in Theorem 1.2 to $p$-quasi-$\mathcal{F}$-groups and $\pi$-quasi-$\mathcal{F}$-groups? 

The following theorem and its corollary give the affirmative answer to this problem.

**Theorem 1.3.** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{R}$ and $\emptyset \neq \pi \subseteq \pi(G)$. Then $G$ is a $\pi$-quasi-$\mathcal{F}$-group if and only if for every $\mathcal{F}$-eccentric $G$-chief factor $H/K$ between $\Phi(F_\pi(G))$ and $F_\pi^*(G)$ of order divisible by at least one prime in $\pi$, the automorphisms of $H/K$ induced by all elements of $G$ are inner.

In particular, if we let $\pi = \{p\}$, then we directly obtain the following corollary:

**Corollary 1.4.** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{R}$ and $p$ a prime. Then $G$ is a $p$-quasi-$\mathcal{F}$-group if and only if for every $\mathcal{F}$-eccentric $G$-chief factor $H/K$ between $\Phi(F_p(G))$ and $F_p^*(G)$ of order divisible by $p$, the automorphisms of $H/K$ induced by all elements of $G$ are inner.

In the above Theorem and Corollary, $F_p(G)$ (respectively, $F_\pi(G)$) is the $p$-Fitting subgroup of $G$ (respectively, the $\pi$-Fitting subgroup of $G$), that is, the maximal normal $p$-nilpotent subgroup of $G$ (respectively, the maximal normal $\pi$-nilpotent subgroup of $G$); $F_p^*(G)$ is the product of all normal $p$-quasinilpotent subgroups of $G$ and $F_\pi^*(G)$ is the product of all normal $\pi$-quasinilpotent subgroups of $G$. By the definitions of $Z_{\pi}(G)$ and $Z_{\pi}(G)$, it is clear that $Z_{\pi}(G) \leq F_p^*(G)$ and $Z_{\pi}(G) \leq F_\pi^*(G)$. Recall that a class $\mathcal{F}$ of groups is called a Fitting formation if $\mathcal{F}$ is both a formation and a Fitting class (see [2, p. 276]). The following three propositions are the main steps of the proof of Theorem 1.3.

**Proposition 1.5.** The class $\mathcal{R}_p^*$ of all $p$-quasinilpotent groups is a Fitting formation.

**Proposition 1.6.** The class $\mathcal{R}_\pi^*$ of all $\pi$-quasinilpotent groups is a Fitting formation.

The following results follow directly from Proposition 1.5 and Proposition 1.6.

**Corollary 1.7.** $F_p^*(G)$ is the largest normal $p$-quasinilpotent subgroup of $G$, and $F_\pi^*(G)$ is the largest normal $\pi$-quasinilpotent subgroup of $G$.

Clearly, $F_p^*(G)$ and $F_\pi^*(G)$ are characteristic subgroups of $G$. We call $F_p^*(G)$ and $F_\pi^*(G)$ the generalized $p$-Fitting subgroup and the generalized $\pi$-Fitting subgroup of $G$, respectively.

**Proposition 1.8.** $F_\pi^*(G/O_{\pi'}(G)) = F_\pi^*(G)/O_{\pi'}(G)$.

In particular, if we let $\pi = \{p\}$, then we directly obtain the following corollary:

**Corollary 1.9.** [1, Lemma 2.10(2)] $F_p^*(G/O_p(G)) = F_p^*(G)/O_p(G)$. 
2. Preliminaries

Recall that a group $G$ is called semisimple if $G$ is the direct product of non-abelian simple groups (see [9, Chap. X, Definition 13.5]). Let $\pi(\mathfrak{F}) = \bigcap_{G \in \mathfrak{F}} \pi(G)$.

**Lemma 2.1.** [3, Chap. I, Theorem 3.7, Theorem 3.12 and Corollary 3.13] Let $\mathfrak{F}$ be a normally hereditary saturated formation containing $\mathfrak{N}$, $p \in \pi(\mathfrak{F})$ and $\emptyset \neq \pi \subseteq \pi(G)$. Then:

1. The classes $\mathfrak{F}^*_p$ and $\mathfrak{F}^*_\pi$ are normally hereditary solubly saturated formations.
2. A group $G$ is a $\pi$-quasi-$\mathfrak{F}$-group if and only if $G/Z_{\pi(\mathfrak{F})}$ is semisimple and the order of each composition factor of $G/Z_{\pi(\mathfrak{F})}$ is divisible by at least one prime $p \in \pi$.

**Lemma 2.2.** ([5, Lemma 2.4] or [3, Chap. I, Lemma 3.5]) Let $H/K$ be a chief factor of $G$. Suppose that the automorphism of $H/K$ induced by an element $g \in G$ is inner. Then $gK \in (H/K)C_{G/K}(H/K)$.

**Lemma 2.3.** ([7, Lemmas 2.13 and 2.14] or [3, Chap. I, Proposition 1.15 and Lemma 2.26]) Let $\mathfrak{F}$ be a saturated (solubly saturated) formation and $f$ the canonical local (the canonical composition, respectively) satellite of $\mathfrak{F}$.

1. A $G$-chief factor $H/K$ is $\mathfrak{F}$-central in $G$ if and only if $G/C_{G}(H/K) \in f(p)$ in the case where $H/K$ is a abelian $p$-group, and $G/C_{G}(H/K) \in \mathfrak{F}$ in the case where $H/K$ is non-abelian.
2. Let $E$ be a normal $p$-subgroup of $G$. Then $E \leq Z_{\mathfrak{F}}(G)$ if and only if $G/C_{G}(E) \in f(p)$.

**Lemma 2.4.** ([3, Chap. I, Proposition 3.6] or [5, Proposition 2.5]) Let $\mathfrak{F} = LF(f)$ a saturated formation, where $f$ is the canonical local satellite of $\mathfrak{F}$ and $\emptyset \neq \pi \subseteq \pi(\mathfrak{F})$. Then the canonical composition satellite $f^*_p$ of $\mathfrak{F}^*_\pi$ satisfies $f^*_p(p) = f(p) = \mathfrak{F}_p f(p) \subseteq \mathfrak{F}$ for all primes $p \in \pi$ and $f^*_p(0) = \mathfrak{F}^*_\pi = f^*_p(p)$ for all primes $p \notin \pi$.

**Lemma 2.5.** ([7, Theorem A(ii)], see also [3, Chap. I, Theorem 2.8(ii)]) Let $\mathfrak{F}$ be any formation and $E$ a normal subgroup of $G$. If $F^*(E) \leq Z_{\mathfrak{F}}(G)$, then $E \leq Z_{\mathfrak{F}}(G)$.

**Lemma 2.6.** [4, Theorems 1.8.20 and 1.8.23]

1. $G$ is $\pi$-nilpotent if and only if $G$ has a normal $\pi$-complement, and any Hall $\pi$-subgroup is nilpotent.
2. $F^*_\pi(G) \subseteq O_{\pi^\prime}(G)$.

3. Proofs of Main Theorems

**Proof of Proposition 1.5.** By Lemma 2.1(1), $\mathfrak{F}^*_p$ is a normally hereditary solubly saturated formation. So we only need to prove that: if $M$ and $N$ are normal $p$-quasinilpotent subgroups of $G$, then $MN$ is also a $p$-quasinilpotent subgroup.
implies that $M/B$ by all elements of $M$. Moreover, $G/L \in \mathfrak{R}_p$.

If $M \cap N = 1$, then $G = M \times N$. It follows that $G$ is $p$-quasinilpotent since $\mathfrak{R}_p$ is a formation. This contradiction shows that $M \cap N > 1$. Let $L$ be a minimal normal subgroup of $G$ contained in $M \cap N$. Then $G/L = (M/L)(N/L)$ and both $M/L$ and $N/L$ are $p$-quasinilpotent. So $G/L$ is $p$-quasinilpotent by the choice of $G$; that is, $G/L \in \mathfrak{R}_p$. If $G$ has another minimal normal subgroup $K$ contained in $M \cap N$, then $L \cap K = 1$ and, analogously, we have $G/K \in \mathfrak{R}_p$. In this case, we have $G \in \mathfrak{R}_p$ for $G \cong G/(L \cap K) \cong G/L \times G/K$. This contradiction shows that $L$ is the unique minimal normal subgroup of $G$ contained in $M \cap N$.

(3) $L$ is a non-abelian group of order divisible by $p$.

If $p \nmid |L|$, then $G$ has a normal series such that all $G$-chief factors in this series of order divisible by $p$ are above $L$. Note that $G/L \in \mathfrak{R}_p$. By the Jordan-H"{o}lder theorem, we have that $G \in \mathfrak{R}_p$, a contradiction. Hence $p \mid |L|$.

Assume that $L$ is abelian and let $A/B$ be any $M$-chief factor below $L$. If $A/B$ is $\mathfrak{R}$-eccentric in $M$, then $M \in \mathfrak{R}_p^*$ implies that the automorphisms of $A/B$ induced by all elements of $M$ are inner. By Lemma 2.2, $M/B = (A/B)C_{M/B}(A/B)$. This implies that $M/B = C_{M/B}(A/B)$, that is, $A/B$ is $\mathfrak{R}$-central in $M$. This contradiction shows that every $M$-chief factor of $L$ is $\mathfrak{R}$-central, so $L \leq Z_\infty(M)$. Note that the canonical local satellite $F$ of $\mathfrak{R}$ satisfies $F(q) = \Phi_q$ for any prime $q$ (see [3, p.3]). Hence $M/C_M(L) \in \Phi_p$ by Lemma 2.3(2). Moreover, $MC_G(L)/C_G(L) \in \Phi_p$ by the $G$-isomorphism $M/C_M(L) = M/(M \cap C_G(L)) \cong MC_G(L)/C_G(L)$. Analogously, $NC_G(L)/C_G(L) \in \Phi_p$. This implies that $G/C_G(L) = MC_G(L)/C_G(L) \cdot NC_G(L)/C_G(L) \in \Phi_p$. Consequently, $L \leq Z_\infty(G)$ by Lemma 2.3(2) again. But since $L$ is a minimal normal subgroup of $G$, we have that $L \leq Z(G)$. Hence the automorphism of $L$ induced by any element of $G$ is the identity. It follows from (2) that $G \in \mathfrak{R}_p$. This contradiction implies that $L$ is non-abelian.

(4) The automorphism of $L$ induced by an elements of $M$ or $N$ is inner.

By (2) and (3), $L = M_1 \times M_2 \times \cdots \times M_s = N_1 \times N_2 \times \cdots \times N_t$, where $M_1, M_2, \cdots, M_s$ are $M$-isomorphism non-abelian minimal normal subgroups of $M$ and $N_1, N_2, \cdots, N_t$ are $N$-isomorphism minimal normal subgroups of $N$ for some integers $s$ and $t$. Also, $p$ divides the order of $M_i$ for $i = 1, 2, \cdots, s$ by (3) again. Obviously, $M_i$ is $\mathfrak{R}$-eccentric in $M$. By the definition, the automorphisms of $M_i$ induced by all elements of $M$ are inner. Hence for every element $m \in M$ and every $i \in \{1, 2, \cdots, s\}$, there exists an element $m_i \in M_i$ such that $x_i^m = x_i^{m_i}$ for any $x_i \in M_i$. Now assume that $(x_1, x_2, \cdots, x_s) \in L$, where $x_i \in M_i$ for $i = 1, 2, \cdots, s$. Then

$$(x_1, x_2, \cdots, x_s)^m = (x_1^m, x_2^m, \cdots, x_s^m) = (x_1^{m_1}, x_2^{m_2}, \cdots, x_s^{m_s}) = (x_1, x_2, \cdots, x_s)^{(m_1, m_2, \cdots, m_s)}.$$
Since \((m_1, m_2, \cdots, m_s) \in M_1 \times M_2 \times \cdots \times M_s = L\), we see that the automorphism of \(L\) induced by \(m \in M\) is inner. This shows that the automorphisms of \(L\) induced by all elements of \(M\) are inner. With a similar argument, we can prove that the automorphisms of \(L\) induced by all elements of \(N\) are inner.

(5) The final contradiction.

For any \(g \in G\), by (1) we may assume that \(g = mn \in G\), for some \(m \in M\) and \(n \in N\). By (4), there exists two elements \(l_1\) and \(l_2\) of \(L\) such that for any \(x \in L\), \(x^m = x^{l_1}\) and \(x^n = x^{l_2}\). Hence \(x^g = x^{mn} = x^{l_1l_2}\) for any \(x \in L\). This shows that the automorphism of \(L\) induced by \(g\) is inner. Now, in view of (2) and the Jordan-Hölder theorem, we obtain that \(G \in \mathfrak{R}_\pi^s\). This contradiction completes the proof.

**Proof of Proposition 1.6.** Clearly, \(\mathfrak{R}_\pi^s = \bigcap_{\mathfrak{R}_p} \mathfrak{R}_\pi^s\). Since the intersection of Fitting formations is a Fitting formation, \(\mathfrak{R}_\pi^s\) is a Fitting formation by the Proposition 1.5

**Proof of Proposition 1.8.** By Corollary 1.7, \(F_\pi^s(G)\) is \(\pi\)-quasinilpotent. Hence from the definition of \(\pi\)-quasinilpotent groups, we see that \(F_\pi^s(G)/O_{\pi'}(G) \leq F_\pi^s(G/O_{\pi'}(G))\). Now let \(F_\pi^s(G/O_{\pi'}(G)) = T/O_{\pi'}(G)\), then \(T/O_{\pi'}(G) \in \mathfrak{R}_\pi^s\) by Corollary 1.7 again. As \(O_{\pi'}(G)\) is a \(\pi'\)-group, \(T \in \mathfrak{R}_\pi^s\) by the Jordan-Hölder theorem. Hence \(F_\pi^s(G/O_{\pi'}(G)) \leq F_\pi^s(G/O_{\pi'}(G))\). It follows that \(F_\pi^s(G/O_{\pi'}(G)) = F_\pi^s(G/O_{\pi'}(G))\).

**Proof of Theorem 1.3.** The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let \(G\) be a counterexample of minimal order. Let \(F = F_\pi(G)\) and \(F^* = F_\pi^s(G)\). We proceed via the following steps.

1. \(F_\pi^s(G/\Phi(F)) = F^*/\Phi(F)\).

By Lemma 2.1(1) and Corollary 1.7, \(F^*/\Phi(F) \leq F_\pi^s(G/\Phi(F))\). Let \(F_\pi^s(G/\Phi(F)) = T/\Phi(F)\). By Lemma 2.6, \(F = N \rtimes H\), where \(N = O_{\pi'}(G)\) and \(H\) a nilpotent Hall \(\pi\)-subgroup of \(F\). By Proposition 1.6 and Corollary 1.7, \((T/N)/(\Phi(F)N/N) \in \mathfrak{R}_\pi^s\).

Since \(\Phi(F)N/N \leq \Phi(F/N)\) (see [2, Chap. A, Theorem 9.2(e)]), \((T/N)/(\Phi(F)N/N) \in \mathfrak{R}_\pi^s\). Note that \(F/N \cong H\) is nilpotent, and so \(H\) is soluble. Hence by Lemma 2.1(1), \(T/N \in \mathfrak{R}_\pi^s\). But as \(N\) is a \(\pi'\)-group, we have \(T \in \mathfrak{R}_\pi^s\) by the Jordan-Hölder theorem. This shows that \(F_\pi^s(G/\Phi(F)) \leq F^*/\Phi(F)\). Thus (1) holds.

2. \(\Phi(F) = 1\), so for every \(\mathfrak{F}\)-eccentric \(G\)-chief factor \(H/K\) below \(F^*\) of order divisible by at least one prime in \(\pi\), every automorphism of \(H/K\) induced by an element of \(G\) is inner.

In view of (1) and the hypothesis, for every \(\mathfrak{F}\)-eccentric \(G/\Phi(F)\)-chief factor \(H/K\) below \(F_\pi^s(G/\Phi(F))\) of order divisible by at least one prime in \(\pi\), every automorphism of \(H/K\) induced by an element of \(G/\Phi(F)\) is inner. This implies that \(G/\Phi(F)\) satisfies the hypothesis. If \(\Phi(F) > 1\), then the choice of \(G\) implies that \(G/\Phi(F) \in \mathfrak{R}_\pi^s\). Using the symbols in (1), \(F = N \rtimes H\). With a similar argument as in (1), we have that \((G/N)/(\Phi(F)N/N) \in \mathfrak{R}_\pi^s\) and \(\Phi(F)N/N \leq \Phi(F/N)\). Hence \((G/N)/(\Phi(F)/N) \in \mathfrak{R}_\pi^s\). Therefore, \(G/N \in \mathfrak{R}_\pi^s\) by the isomorphism \(F/N \cong H\) and
Lemma 2.1(1). Note that $N$ is $\pi'$-group. By the Jordan-Hölder theorem, $G \in \mathfrak{F}$. This contradiction shows that $\Phi(F) = 1$.

(3) $O_{\pi'}(G) = 1$, so $F_{\pi'}(G) \subseteq O_{\pi'}(G)$.

Assume that $O_{\pi'}(G) > 1$. By Proposition 1.8, $F_{\pi'}^*(G/O_{\pi'}(G)) = F_{\pi'}^*/O_{\pi'}(G)$. Thus by (2), for every $\mathfrak{F}$-eccentric $G/O_{\pi'}(G)$-chief factor $H/K$ of $F_{\pi'}^*(G/O_{\pi'}(G))$ of order divisible by at least one prime in $\pi$, every automorphism of $H/K$ induced by an element of $G/O_{\pi'}(G)$ is inner. This implies that $G/O_{\pi'}(G)$ satisfies the hypothesis for $G$. The choice of $G$ implies that $G/O_{\pi'}(G) \in \mathfrak{F}$ and consequently $G \in \mathfrak{F}$. This contradiction shows that $O_{\pi'}(G) = 1$. Thus by Lemma 2.6(2), $F_{\pi'}(G) \subseteq O_{\pi'}(G)$.

(4) The order of each $G$-chief factor below $F^*$ is divisible by at least one prime in $\pi$.

By Lemma 2.1(2) and Corollary 1.7, $F^*/\mathfrak{F}(F^*)$ is semisimple and the order of each composition factor of $F^*/\mathfrak{F}(F^*)$ is divisible by at least prime in $\pi$. Since $Z_{\pi\mathfrak{F}}(F^*) \leq F_{\pi'}(F^*) \leq O_{\pi'}(F^*) \leq O_{\pi'}(G)$ by (3), $F^*/O_{\pi'}(G)$ is semisimple and the order of each composition factor of $F^*/O_{\pi'}(G)$ is divisible by at least one prime in $\pi$. Hence we have (4).

(5) $F^* \leq Z_{\mathfrak{F}}(G)$, which gives the final contradiction.

By Lemmas 2.1(1) and 2.4, $\mathfrak{F}$ is a solubly saturated formation and so it has the unique canonical composition satellite $f_{\pi'}^*$ satisfying $f_{\pi'}^*(p) = f(p) = \mathfrak{F}_p f(p) \subseteq \mathfrak{F}$ for all primes $p \in \pi$ and $f_{\pi'}^*(0) = \mathfrak{F}_{\pi'} = f_{\pi'}^*(p)$ for all primes $p \notin \pi$. Let $H/K$ be a $G$-chief factor below $F^*$. Assume that $H/K$ is $\mathfrak{F}$-eccentric in $G$. Then by (4) and the hypothesis, every automorphism of $H/K$ induced by an element of $G$ is inner. Hence by Lemma 2.2, $G/K = (H/K)G_{G/K}(H/K)$. If $H/K$ is abelian, then it is a $p$-group for some $p \in \pi$ by (4), and $G/K = G_{G/K}(H/K)$. This implies that $G/G_{G}(H/K) \cong (G/K)/G_{G/K}(H/K) = 1 \in f_{\pi'}^*(p)$. Now assume that $H/K$ is non-abelian. Then $G/K = H/K \times G_{G/K}(H/K)$ and $H/K$ is a direct product of some isomorphism non-abelian simple groups. In this case, $H/K$ is semisimple. Hence $H/K$ is quasinilpotent (see [9, p.125]), and so $H/K \in \mathfrak{F} \subseteq \mathfrak{F}$. Therefore $G/G_{G}(H/K) \in \mathfrak{F} = f_{\pi'}^*(0)$ by the $G$-isomorphism $G/G_{G}(H/K) \cong (G/K)/G_{G/K}(H/K) \cong H/K$. Combining with Lemma 2.3(1), the above shows that $H/K$ is $\mathfrak{F}$-central in $G$. Hence $F^* \leq Z_{\mathfrak{F}}(G)$. But since $F^*(G) \leq F_{\pi'}^*(G)$, we obtain that $F^*(G) \leq Z_{\mathfrak{F}}(G)$. It follows from by Lemma 2.5 and 2.1(1) that $G \leq Z_{\mathfrak{F}}(G)$. Consequently, $G \in \mathfrak{F}$. The final contradiction completes the proof.

Acknowledgements. The authors cordially thank the referees for their helpful comments.

4. References

REFERENCES


