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On an open problem about π-quasi-$\mathfrak{F}$-groups

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Abstract – Let $\mathfrak{F}$ be a class of finite groups, p a prime and π a set of some primes. A finite group G is called a p-quasi-$\mathfrak{F}$-group (respectively, by π-quasi-$\mathfrak{F}$-group) provided that for every $\mathfrak{F}$-eccentric G-chief factor H/K of order divisible by p (respectively, by at least one prime in π), the automorphisms of H/K induced by all elements of G are inner. In this paper, we obtain the characterizations of p-quasi-$\mathfrak{F}$-groups and π-quasi-$\mathfrak{F}$-groups, which give a positive answer to an open problem in the book [3].

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1. Introduction

Throughout this paper, all groups are finite. G always denotes a group, p is a prime and $\pi(G)$ is the set of all prime divisors of $|G|$. As usual, we use $\mathfrak{G}_p$ and $\mathfrak{N}$ to denote the class of all p-groups and the class of all nilpotent groups, respectively. All unexplained notation and terminology are standard, as in [2, 3, 9].

Recall that a class $\mathfrak{F}$ of groups is called a formation if $\mathfrak{F}$ is closed under taking homomorphic images and subdirect products. A formation $\mathfrak{F}$ is said to be:

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(1) saturated (solubly saturated) if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$ (respectively, $G/\Phi(N) \in \mathfrak{F}$ for some soluble normal subgroup N of G); (2) hereditary (normally hereditary) if $H \in \mathfrak{F}$ whenever $H \leq G \in \mathfrak{F}$ (respectively, $H \trianglelefteq G \in \mathfrak{F}$). It is well known that a G-chief factor H/K is called $\mathfrak{F}$-central in G provided that $H/K \rtimes G/C_G(H/K) \in \mathfrak{F}$. Otherwise, it is called $\mathfrak{F}$-eccentric. Following [8], a normal subgroup N of G is said to be $p\mathfrak{F}$-hypercentral (respectively, $\pi\mathfrak{F}$-hypercentral) in G if either $N = 1$ or $N \neq 1$ and every G-chief factor below N of order divisible by p (respectively, by at least one prime in π) is $\mathfrak{F}$-central in G. The symbol $Z_{p\mathfrak{F}}(G)$ denotes the $p\mathfrak{F}$-hypercentre of G, that is, the product of all normal $p\mathfrak{F}$-hypercentral subgroups of G, and the symbol $Z_{\pi\mathfrak{F}}(G)$ denotes the $\pi\mathfrak{F}$-hypercentre of G, that is, the product of all normal $\pi\mathfrak{F}$-hypercentral subgroups of G. Also, the $\mathfrak{F}$-hypercentre of G denoted by $Z_{\mathfrak{F}}(G)$ (see [2, p.389]), is the product of all normal subgroups N of G, where N is $p\mathfrak{F}$-hypercentral in G for every prime p. In particular, if $\mathfrak{F} = \mathfrak{N}$, then $Z_{\mathfrak{N}}(G)$ is the hypercentre of G and is often denoted by $Z_\infty(G)$.

A group G is said to be quasinilpotent [9, Chap. X, Definition 13.2] if for every G-chief factor H/K and $x \in G$, x induces an inner automorphism on H/K. A group G is said to be p-quasinilpotent [10] if for every G-chief factor R/L of order divisible by p and $x \in G$, x induces an inner automorphism on R/L.

As a important generalization of quasinilpotent groups and p-quasinilpotent groups, Guo and Skiba [5, 6] introduced the following notion:

Definition 1.1. [5, 6] (see also [3, Chap. 1, Definition 3.2]) Let $\mathfrak{F}$ be a class of groups, G a group and p a prime. Assume that $\emptyset \neq \pi \subseteq \pi(G)$. G is called a p-quasi-$\mathfrak{F}$-group (respectively, π-quasi-$\mathfrak{F}$-group) if for every $\mathfrak{F}$-eccentric G-chief factor H/K of order divisible by p (respectively, at least one prime in π), the automorphisms of H/K induced by all elements of G are inner. In particular, if $\pi = \mathbb{P}$ is the set of all primes, then a π-quasi-$\mathfrak{F}$-group is called a quasi-$\mathfrak{F}$-group.

Following [5, 3], we use $\mathfrak{F}^*$, $\mathfrak{F}^*_p$ and $\mathfrak{F}^*_\pi$ to denote the class of all quasi-$\mathfrak{F}$-groups, p-quasi-$\mathfrak{F}$-groups and π-quasi-$\mathfrak{F}$-groups, respectively. Clearly, $\mathfrak{F}^* \subseteq \mathfrak{F}^*_\pi$, and $\mathfrak{F}^*_\pi = \bigcap_{p \in \pi} \mathfrak{F}^*_p$. For the details, one can refer to [3, Chap. I, §1.3].

In [5], Guo and Skiba have given the general theory of the quasi-$\mathfrak{F}$-groups and obtained some characterizations of quasisoluble groups and quasisupersoluble groups. In particular, they proved the following theorem.

Theorem 1.2. [5, Theorem B] *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{N}$. Then G is a quasi-$\mathfrak{F}$-group if and only if for every $\mathfrak{F}$-eccentric G-chief factor H/K between $\Phi(F(G))$ and $F^*(G)$, the automorphisms of H/K induced by all elements of G are inner.*

Here, $F^*(G)$ is the generalized Fitting subgroup of G (see [9, Chap. X, Definition 13.9]) , that is, the set of all elements of G which induce inner automorphisms on every G-chief factor. Clearly, $F^*(G)$ is a characteristic subgroup of G.

In connection with this, the author in the book [3] proposed the following open problem:

Problem. [3, Chap. I, Problem 6.1] *Could we generalize the result in Theorem 1.2 to p-quasi-$\mathfrak{F}$-groups and π-quasi-$\mathfrak{F}$-groups ?*

The following theorem and its corollary give the affirmative answer to this problem.

Theorem 1.3. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{N}$ and $\emptyset \neq \pi \subseteq \pi(G)$. Then G is a π-quasi-$\mathfrak{F}$-group if and only if for every $\mathfrak{F}$-eccentric G-chief factor H/K between $\Phi(F_\pi(G))$ and $F^*_\pi(G)$ of order divisible by at least one prime in π, the automorphisms of H/K induced by all elements of G are inner.*

In particular, if we let $\pi = \{p\}$, then we directly obtain the following corollary:

Corollary 1.4. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{N}$ and p a prime. Then G is a p-quasi-$\mathfrak{F}$-group if and only if for every $\mathfrak{F}$-eccentric G-chief factor H/K between $\Phi(F_p(G))$ and $F^*_p(G)$ of order divisible by p, the automorphisms of H/K induced by all elements of G are inner.*

In the above Theorem and Corollary, $F_p(G)$ (respectively, $F_\pi(G)$) is the p-Fitting subgroup of G (respectively, the π-Fitting subgroup of G), that is, the maximal normal p-nilpotent subgroup of G (respectively, the maximal normal π-nilpotent subgroup of G); $F^*_p(G)$ is the product of all normal p-quasinilpotent subgroups of G and $F^*_\pi(G)$ is the product of all normal π-quasinilpotent subgroups of G. By the definitions of $Z_{p\mathfrak{N}}(G)$ and $Z_{\pi\mathfrak{N}}(G)$, it is clear that $Z_{p\mathfrak{N}}(G) \leq F^*_p(G)$ and $Z_{\pi\mathfrak{N}}(G) \leq F^*_\pi(G)$. Recall that a class $\mathfrak{F}$ of groups is called a Fitting formation if $\mathfrak{F}$ is both a formation and a Fitting class (see [2, p. 276]). The following three propositions are the main steps of the proof of Theorem 1.3.

Proposition 1.5. *The class $\mathfrak{N}^*_p$ of all p-quasinilpotent groups is a Fitting formation.*

Proposition 1.6. *The class $\mathfrak{N}^*_\pi$ of all π-quasinilpotent groups is a Fitting formation.*

The following results follow directly from Proposition 1.5 and Proposition 1.6.

Corollary 1.7. *$F^*_p(G)$ is the largest normal p-quasinilpotent subgroup of G, and $F^*_\pi(G)$ is the largest normal π-quasinilpotent subgroup of G.*

Clearly, $F^*_p(G)$ and $F^*_\pi(G)$ are characteristic subgroups of G. We call $F^*_p(G)$ and $F^*_\pi(G)$ the generalized p-Fitting subgroup and the generalized π-Fitting subgroup of G, respectively.

Proposition 1.8. $F^*_\pi(G/O_{\pi'}(G)) = F^*_\pi(G)/O_{\pi'}(G)$.

In particular, if we let $\pi = \{p\}$, then we directly obtain the following corollary:

Corollary 1.9. [1, Lemma 2.10(2)] $F^*_p(G/O_{p'}(G)) = F^*_p(G)/O_{p'}(G)$.

2. Preliminaries

Recall that a group G is called semisimple if G is the direct product of non-abelian simple groups (see [9, Chap. X, Definition 13.5]). Let $\pi(\mathfrak{F}) = \bigcap_{G\in\mathfrak{F}} \pi(G)$.

LEMMA 2.1. [3, Chap. I, Theorem 3.7, Theorem 3.12 and Corollary 3.13] *Let $\mathfrak{F}$ be a normally hereditary saturated formation containing $\mathfrak{N}$, $p \in \pi(\mathfrak{F})$ and $\emptyset \neq \pi \subseteq \pi(G)$. Then:*

(1) *The classes $\mathfrak{F}_p^*$ and $\mathfrak{F}_\pi^*$ are normally hereditary solubly saturated formations.*

(2) *A group G is a π-quasi-$\mathfrak{F}$-group if and only if $G/Z_{\pi\mathfrak{F}}(G)$ is semisimple and the order of each composition factor of $G/Z_{\pi\mathfrak{F}}(G)$ is divisible by at least one prime $p \in \pi$.*

LEMMA 2.2. ([5, Lemma 2.4] or [3, Chap. I, Lemma 3.5]) *Let H/K be a chief factor of G. Suppose that the automorphism of H/K induced by an element $g \in G$ is inner. Then $gK \in (H/K)C_{G/K}(H/K)$.*

LEMMA 2.3. ([7, Lemmas 2.13 and 2.14] or [3, Chap. I, Proposition 1.15 and Lemma 2.26]) *Let $\mathfrak{F}$ be a saturated (solubly saturated) formation and f the canonical local (the canonical composition, respectively) satellite of $\mathfrak{F}$.*

(1) *A G-chief factor H/K is $\mathfrak{F}$-central in G if and only if $G/C_G(H/K) \in f(p)$ in the case where H/K is a abelian p-group, and $G/C_G(H/K) \in \mathfrak{F}$ in the case where H/K is non-abelian.*

(2) *Let E be a normal p-subgroup of G. Then $E \leq Z_{\mathfrak{F}}(G)$ if and only if $G/C_G(E) \in f(p)$.*

LEMMA 2.4. ([3, Chap. I, Proposition 3.6] or [5, Proposition 2.5]) *Let $\mathfrak{F} = LF(f)$ a saturated formation, where f is the canonical local satellite of $\mathfrak{F}$ and $\emptyset \neq \pi \subseteq \pi(\mathfrak{F})$. Then the canonical composition satellite f_π^* of $\mathfrak{F}_\pi^*$ satisfies $f_\pi^*(p) = f(p) = \mathfrak{G}_p f(p) \subseteq \mathfrak{F}$ for all primes $p \in \pi$ and $f_\pi^*(0) = \mathfrak{F}_\pi^* = f_\pi^*(p)$ for all primes $p \notin \pi$.*

LEMMA 2.5. ([7, Theorem A(ii)], see also [3, Chap. I, Theorem 2.8(ii)]) *Let $\mathfrak{F}$ be any formation and E a normal subgroup of G. If $F^*(E) \leq Z_{\mathfrak{F}}(G)$, then $E \leq Z_{\mathfrak{F}}(G)$.*

LEMMA 2.6. [4, Theorems 1.8.20 and 1.8.23]

(1) *G is π-nilpotent if and only if G has a normal π-complement, and any Hall π-subgroup is nilpotent.*

(2) *$F_\pi(G) \subseteq O_{\pi\pi'}(G)$.*

3. Proofs of Main Theorems

Proof of Proposition 1.5. By Lemma 2.1(1), $\mathfrak{N}_p^*$ is a normally hereditary solubly saturated formation. So we only need to prove that: *if M and N are normal p-quasinilpotent subgroups of G, then MN is also a p-quasinilpotent subgroup* *of* G. Assume that the assertion is false and let G be a counterexample of minimal order. We proceed via the following steps.

(1) $G = MN$.

Assume that $MN < G$. Then the choice of G implies that MN is p-quasinilpotent, a contradiction. Thus (1) holds.

(2) $M \cap N > 1$ *and there exists a unique minimal normal subgroup* L *of* G *contained in* $M \cap N$. *Moreover,* $G/L \in \mathfrak{N}_p^*$.

If $M \cap N = 1$, then $G = M \times N$. It follows that G is p-quasinilpotent since $\mathfrak{N}_p^*$ is a formation. This contradiction shows that $M \cap N > 1$. Let L be a minimal normal subgroup of G contained in $M \cap N$. Then $G/L = (M/L)(N/L)$ and both M/L and N/L are p-quasinilpotent. So G/L is p-quasinilpotent by the choice of G, that is, $G/L \in \mathfrak{N}_p^*$. If G has another minimal normal subgroup K contained in $M \cap N$, then $L \cap K = 1$ and, analogously, we have $G/K \in \mathfrak{N}_p^*$. In this case, we have $G \in \mathfrak{N}_p^*$ for $G \cong G/(L \cap K) \cong G/L \times G/K$. This contradiction shows that L is the unique minimal normal subgroup of G contained in $M \cap N$.

(3) L *is a non-abelian group of order divisible by* p.

If $p \nmid |L|$, then G has a normal series such that all G-chief factors in this series of order divisible by p are above L. Note that $G/L \in \mathfrak{N}_p^*$. By the Jordan-Hölder theorem, we have that $G \in \mathfrak{N}_p^*$, a contradiction. Hence $p \mid |L|$.

Assume that L is abelian and let A/B be any M-chief factor below L. If A/B is $\mathfrak{N}$-eccentric in M, then $M \in \mathfrak{N}_p^*$ implies that the automorphisms of A/B induced by all elements of M are inner. By Lemma 2.2, $M/B = (A/B)C_{M/B}(A/B)$. This implies that $M/B = C_{M/B}(A/B)$, that is, A/B is $\mathfrak{N}$-central in M. This contradiction shows that every M-chief factor of L is $\mathfrak{N}$-central, so $L \leq Z_\infty(M)$. Note that the canonical local satellite F of $\mathfrak{N}$ satisfies $F(q) = \mathfrak{G}_q$ for any prime q (see [3, p.3]). Hence $M/C_M(L) \in \mathfrak{G}_p$ by Lemma 2.3(2). Moreover, $MC_G(L)/C_G(L) \in \mathfrak{G}_p$ by the G-isomorphism $M/C_M(L) = M/(M \cap C_G(L)) \cong MC_G(L)/C_G(L)$. Analogously, $NC_G(L)/C_G(L) \in \mathfrak{G}_p$. This implies that $G/C_G(L) = MC_G(L)/C_G(L) \cdot NC_G(L)/C_G(L) \in \mathfrak{G}_p$. Consequently, $L \leq Z_\infty(G)$ by Lemma 2.3(2) again. But since L is a minimal normal subgroup of G, we have that $L \leq Z(G)$. Hence the automorphism of L induced by any element of G is the identity. It follows from (2) that $G \in \mathfrak{N}_p^*$. This contradiction implies that L is non-abelian.

(4) *The automorphism of* L *induced by an elements of* M *or* N *is inner.*

By (2) and (3), $L = M_1 \times M_2 \times \cdots \times M_s = N_1 \times N_2 \times \cdots \times N_t$, where $M_1, M_2, \cdots, M_s$ are M-isomorphism non-abelian minimal normal subgroups of M and $N_1, N_2, \cdots, N_t$ are N-isomorphism minimal normal subgroups of N for some integers s and t. Also, p divides the order of M_i for $i = 1, 2, \cdots s$ by (3) again. Obviously, M_i is $\mathfrak{N}$-eccentric in M. By the definition, the automorphisms of M_i induced by all elements of M are inner. Hence for every element $m \in M$ and every $i \in \{1, 2, \cdots, s\}$, there exists an element $m_i \in M_i$ such that $x_i^m = x_i^{m_i}$ for any $x_i \in M_i$. Now assume that $(x_1, x_2, \cdots, x_s) \in L$, where $x_i \in M_i$ for $i = 1, 2, \cdots, s$. Then

$$(x_1, x_2, \cdots, x_s)^m = (x_1^m, x_2^m, \cdots, x_s^m) = (x_1^{m_1}, x_2^{m_2}, \cdots, x_s^{m_s}) = (x_1, x_2, \cdots, x_s)^{(m_1, m_2, \cdots, m_s)}.$$

Since $(m_1, m_2, \cdots, m_s) \in M_1 \times M_2 \times \cdots \times M_s = L$, we see that the automorphism of L induced by $m \in M$ is inner. This shows that the automorphisms of L induced by all elements of M are inner. With a similar argument, we can prove that the automorphisms of L induced by all elements of N are inner.

(5) *The final contradiction.*

For any $g \in G$, by (1) we may assume that $g = mn \in G$, for some $m \in M$ and $n \in N$. By (4), there exists two elements l_1 and l_2 of L such that for any $x \in L$, $x^m = x^{l_1}$ and $x^n = x^{l_2}$. Hence $x^g = x^{mn} = x^{l_1 l_2}$ for any $x \in L$. This show that the automorphism of L induced by g is inner. Now, in view of (2) and the Jordan-Hölder theorem, we obtain that $G \in \mathfrak{N}_p^*$. This contradiction completes the proof.

Proof of Proposition 1.6. Clearly, $\mathfrak{N}_\pi^* = \bigcap_{p\in\pi} \mathfrak{N}_p^*$. Since the intersection of Fitting formations is a Fitting formation, $\mathfrak{N}_\pi^*$ is a Fitting formation by the Proposition 1.5

Proof of Proposition 1.8. By Corollary 1.7, $F_\pi^*(G)$ is π-quasinilpotent. Hence from the definition of π-quasinilpotent groups, we see that $F_\pi^*(G)/O_{\pi'}(G) \leq F_\pi^*(G/O_{\pi'}(G))$. Now let $F_\pi^*(G/O_{\pi'}(G)) = T/O_{\pi'}(G)$, then $T/O_{\pi'}(G) \in \mathfrak{N}_\pi^*$ by Corollary 1.7 again. As $O_{\pi'}(G)$ is a π'-group, $T \in \mathfrak{N}_\pi^*$ by the Jordan-Hölder theorem. Hence $F_\pi^*(G/O_{\pi'}(G)) \leq F_\pi^*(G)/O_{\pi'}(G)$. It follows that $F_\pi^*(G/O_{\pi'}(G)) = F_\pi^*(G)/O_{\pi'}(G)$.

Proof of Theorem 1.3. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order. Let $F = F_\pi(G)$ and $F^* = F_\pi^*(G)$. We proceed via the following steps.

(1) $F_\pi^*(G/\Phi(F)) = F^*/\Phi(F)$.

By Lemma 2.1(1) and Corollary 1.7, $F^*/\Phi(F) \leq F_\pi^*(G/\Phi(F))$. Let $F_\pi^*(G/\Phi(F)) = T/\Phi(F)$. By Lemma 2.6, $F = N \rtimes H$, where $N = O_{\pi'}(G)$ and H a nilpotent Hall π-subgroup of F. By Proposition 1.6 and Corollary 1.7, $(T/N)/(\Phi(F)N/N) \in \mathfrak{N}_\pi^*$. Since $\Phi(F)N/N \leq \Phi(F/N)$ (see [2, Chap. A, Theorem 9.2(e)]), $(T/N)/\Phi(F/N) \in \mathfrak{N}_\pi^*$. Note that $F/N \cong H$ is nilpotent, and so H is soluble. Hence by Lemma 2.1(1), $T/N \in \mathfrak{N}_\pi^*$. But as N is a π'-group, we have $T \in \mathfrak{N}_p^*$ by the Jordan-Hölder theorem. This shows that $F_\pi^*(G/\Phi(F)) \leq F^*/\Phi(F)$. Thus (1) holds.

(2) $\Phi(F) = 1$, *so for every $\mathfrak{F}$-eccentric G-chief factor H/K below F^* of order divisible by at least one prime in π, every automorphism of H/K induced by an element of G is inner.*

In view of (1) and the hypothesis, for every $\mathfrak{F}$-eccentric $G/\Phi(F)$-chief factor H/K below $F_\pi^*(G/\Phi(F))$ of order divisible by at least one prime in π, every automorphism of H/K induced by an element of $G/\Phi(F)$ is inner. This implies that $G/\Phi(F)$ satisfies the hypothesis. If $\Phi(F) > 1$, then the choice of G implies that $G/\Phi(F) \in \mathfrak{F}_\pi^*$. Using the symbols in (1), $F = N \rtimes H$. With a similar argument as in (1), we have that $(G/N)/(\Phi(F)N/N) \in \mathfrak{F}_\pi^*$ and $\Phi(F)N/N \leq \Phi(F/N)$. Hence $(G/N)/\Phi(F/N) \in \mathfrak{F}_\pi^*$. Therefore, $G/N \in \mathfrak{F}_\pi^*$ by the isomorphism $F/N \cong H$ and

Lemma 2.1(1). Note that N is π'-group. By the Jordan-Hölder theorem, $G \in \mathfrak{F}_\pi^*$. This contradiction shows that $\Phi(F) = 1$.

(3) *$O_{\pi'}(G) = 1$, so $F_\pi(G) \subseteq O_\pi(G)$.*

Assume that $O_{\pi'}(G) > 1$. By Proposition 1.8, $F_\pi^*(G/O_{\pi'}(G)) = F^*/O_{\pi'}(G)$. Thus by (2), for every $\mathfrak{F}$-eccentric $G/O_{\pi'}(G)$-chief factor H/K of $F_\pi^*(G/O_{\pi'}(G))$ of order divisible by at least one prime in π, every automorphism of H/K induced by an element of $G/O_{\pi'}(G)$ is inner. This implies that $G/O_{\pi'}(G)$ satisfies the hypothesis for G. The choice of G implies that $G/O_{\pi'}(G) \in \mathfrak{F}_\pi^*$ and consequently $G \in \mathfrak{F}_\pi^*$. This contradiction shows that $O_{\pi'}(G) = 1$. Thus by Lemma 2.6(2), $F_\pi(G) \subseteq O_\pi(G)$.

(4) *The order of each G-chief factor below F^* is divisible by at least one prime in π.*

By Lemma 2.1(2) and Corollary 1.7, $F^*/Z_{\pi\mathfrak{N}}(F^*)$ is semisimple and the order of each composition factor of $F^*/Z_{\pi\mathfrak{N}}(F^*)$ is divisible by at least prime in π. Since $Z_{\pi\mathfrak{N}}(F^*) \leq F_\pi(F^*) \leq O_\pi(F^*) \leq O_\pi(G)$ by (3), $F^*/O_\pi(G)$ is semisimple and the order of each composition factor of $F^*/O_\pi(G)$ is divisible by at least one prime in π. Hence we have (4).

(5) *$F^* \leq Z_{\mathfrak{F}_\pi^*}(G)$, which gives the final contradiction.*

By Lemmas 2.1(1) and 2.4, $\mathfrak{F}_\pi^*$ is a solubly saturated formation and so it has the unique canonical composition satellite f_π^* satisfying $f_\pi^*(p) = f(p) = \mathfrak{G}_p f(p) \subseteq \mathfrak{F}$ for all primes $p \in \pi$ and $f_\pi^*(0) = \mathfrak{F}_\pi^* = f_\pi^*(p)$ for all primes $p \notin \pi$. Let H/K be a G-chief factor below F^*. Assume that H/K is $\mathfrak{F}$-eccentric in G. Then by (4) and the hypothesis, every automorphism of H/K induced by an element of G is inner. Hence by Lemma 2.2, $G/K = (H/K)G_{G/K}(H/K)$. If H/K is abelian, then it is a p-group for some $p \in \pi$ by (4), and $G/K = G_{G/K}(H/K)$. This implies that $G/G_G(H/K) \cong (G/K)/G_{G/K}(H/K) = 1 \in f_\pi^*(p)$. Now assume that H/K is non-abelian. Then $G/K = H/K \times G_{G/K}(H/K)$ and H/K is a direct product of some isomorphism non-abelian simple groups. In this case, H/K is semisimple. Hence H/K is quasinilpotent (see [9, p.125]), and so $H/K \in \mathfrak{N}^* \subseteq \mathfrak{F}_\pi^*$. Therefore $G/G_G(H/K) \in \mathfrak{F}_\pi^* = f_\pi^*(0)$ by the G-isomorphism $G/G_G(H/K) \cong (G/K)/G_{G/K}(H/K) \cong H/K$. Combining with Lemma 2.3(1), the above shows that H/K is $\mathfrak{F}_\pi^*$-central in G. Hence $F^* \leq Z_{\mathfrak{F}_\pi^*}(G)$. But since $F^*(G) \leq F_\pi^*(G)$, we obtain that $F^*(G) \leq Z_{\mathfrak{F}_\pi^*}(G)$. It follows from by Lemma 2.5 and 2.1(1) that $G \leq Z_{\mathfrak{F}_\pi^*}(G)$. Consequently, $G \in \mathfrak{F}_\pi^*$. The final contradiction completes the proof.

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