Addendum to «Prime graph components of finite almost simple groups».

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This is an addendum to the paper [2]. We refer to that paper for the definitions. We denote by $G(G)$ the prime graph of a finite group $G$.

The following groups have to be added to the list of the almost simple groups in which the prime graph is not connected in [2].

**Proposition 1.** Let $G$ be one of the following almost simple groups:
- $S < G \leq \text{Aut}(S)$ and $G$ does not split over $S$, with $S$ a simple non abelian group such that $\Gamma(S)$ is not connected;
- $\text{PGL}(2, q)$, with $q = p^s$, $p$ an odd prime;
- $G = Sz(q)(\alpha)$ where $q = 2^f$, $f$ is an odd prime number and $\alpha$ is a field automorphism of order $f$;
- $G = \text{Ree}(q)(\alpha)$ where $q = 3^f$, $f$ is an odd prime number and $\alpha$ is a field automorphism of order $f$;
- $\text{PSL}(3, 4) = S < G < \text{Aut}(S)$ and $|G:S| = 2$.

Then $\Gamma(G)$ is not connected.

**Proof.** Let $G$ be an almost simple group $S < G \leq \text{Aut}(S)$ such that $G$ does not split, we prove that $\Gamma(G) = \Gamma(S)$. As at the beginning of the proof of the case of Lie groups in [2], we can suppose that $|G:S| = p$. Let $x$ be an element of $G\setminus S$, then $x^p \in S$ but $x^p \neq 1$. Moreover we can suppose that $x$ is a $p$-element. Then $C_p(x) < C_p(x^p)$, therefore for any prime $r \in \pi(G)$ we have that $r - p$ in $\Gamma(G)$ if and only if $r - p$ in $\Gamma(S)$.

If $G = \text{PGL}(2, q)$ with $q = p^f$, $p$ an odd prime, then $\pi_1(G) = \pi(q^2 - 1)$ and $\pi_2(G) = \{p\}$.

Let $G = Sz(q)(\alpha)$ where $q = 2^f$, $f$ is an odd prime number and $\alpha$ is a field automorphism of order $f$. If $S = Sz(q) = B_2(q)$, then the connected

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components of $\Gamma(S)$ not containing 2 are

$$\pi_2(S) = \pi(q - 1), \quad \pi_3(S) = \pi(q - \sqrt{2q} + 1), \quad \pi_4(S) = \pi(q + \sqrt{2q} + 1).$$

If $\alpha$ is a field automorphism, then $\pi(C_2(\alpha)) = \pi(\sqrt{2}B_2(2)) = \{2, 5\}$ (see [1], Theorem 9.1). We observe that 5 divides $2^{2^2} + 1$. It can be proved that, if $f \equiv 1, 7 (8)$, 5 divides $(q + \sqrt{2q} + 1)$; or, if $f \equiv 3, 5 (8)$, then 5 divides $(q - \sqrt{2q} + 1)$. Therefore

$$\pi_1(G) = \{2, f\} \cup \pi_3(S) \quad \pi_2(G) = \pi_2(S) \quad \pi_3(G) = \pi_3(S) \quad \text{if} \ f \equiv 1, 7 (8),$$

$$\pi_1(G) = \{2, f\} \cup \pi_3(S) \quad \pi_2(G) = \pi_2(S) \quad \pi_3(G) = \pi_4(S) \quad \text{if} \ f \equiv 3, 5 (8).$$

Let $G = \text{Ree}(q)/\alpha$ where $q = 3^f$, $f$ is an odd prime number and $\gamma$ is a field automorphism of order $f$. If $S = \text{Ree}(q) = ^2G_2(q)$, then the connected components of $\Gamma(S)$ not containing 2 are

$$\pi_2(S) = \pi(q - \sqrt{3q} + 1), \quad \pi_3(S) = \pi(q + \sqrt{3q} + 1).$$

If $\alpha$ is a field automorphism, then $\pi(C_3(\alpha)) = \pi(\sqrt{3}G_3(3)) = \{2, 3, 7\}$ (see [1], Theorem 9.1). We observe that 7 divides $3^{2^2} + 1$. It can be proved that, if $f \equiv 1, 11 (12)$, 7 divides $(q + \sqrt{3q} + 1)$; or, if $f \equiv 5, 7 (12)$, then 7 divides $(q - \sqrt{3q} + 1)$. Therefore

$$\pi_1(G) = \pi(q(q^2 - 1) f) \cup \pi_3(S) \quad \pi_2(G) = \pi_2(S) \quad \text{if} \ f \equiv 1, 11 (12),$$

$$\pi_1(G) = \pi(q(q^2 - 1) f) \cup \pi_3(S) \quad \pi_2(G) = \pi_3(S) \quad \text{if} \ f \equiv 5, 7 (12).$$

If $S = \text{PSL}(3, 4)$ and $G \leq \text{Aut}(S)$ with $|G/S| = 2$, then we have the following three possibilities:

$G = S.2_1$, extension with a graph-field automorphism, then $\pi_1(G) = \{2, 3\}$, $\pi_3(G) = \{5\}$ and $\pi_2(G) = \{7\}$.

$G = S.2_2$, extension with a field automorphism, then $\pi_1(G) = \{2, 3, 7\}$, $\pi_2(G) = \{5\}$.

$G = S.2_3$, extension with a graph automorphism, then $\pi_1(G) = \{2, 3, 5\}$, $\pi_2(G) = \{7\}$.

REFERENCES

