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Abstract - A uniaxially-inhomogeneous layer is considered, sandwiched between two homogeneous half spaces. The material of both layer and half spaces is a linearly elastic and isotropic solid. Reflection and transmission are regarded as originated by a normally-incident transient wave. The new idea exploited in this paper consists in writing the boundary conditions for the layer in a way which accounts directly for the outgoing character of the unknown waves. Uniqueness theorems are given and the solution is determined through Fourier analysis. It follows that in general the solution is a convolution of the incident wave function.

1. Introduction.

The mathematical modeling of wave reflection and transmission through a region presents a peculiar difficulty: at least on a part of the boundary, both the incident and the reflected/transmitted waves occur simultaneously; therefore, since we cannot pick a surface where the solution is known, the problem is not a standard initial-boundary value problem. This explains why ordinarily existence and/or uniqueness results are lacking in reflection-transmission problems.

For definiteness and simplicity, in this paper we restrict attention to normal incidence on a layer sandwiched between two homogeneous half spaces. The layer is uniaxially inhomogeneous, in that the material prop-

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erties depend on a single coordinate. In addition, we let the material be linear, elastic and isotropic. An analogous procedure can be applied to electromagnetic waves in uniaxially inhomogeneous media.

In any reflection-transmission or scattering process, the unknown waves are all outgoing, namely, they propagate away from the scatterer (the layer). The key idea we exploit in this paper consists in writing the boundary conditions for the layer in a way which accounts directly for the outgoing character of the unknown waves. Also, we adopt a condition at space infinity which merely reflects the finiteness of the speed of propagation. We are then able to prove uniqueness of $C^2$ solutions in the whole space domain through an energy method. As to existence, we look for solutions through Fourier analysis which allows us to describe processes by a simple first-order system of ordinary equations. We find that, in general, the reflected and transmitted parts of the solution are appropriate convolutions of the incident wave. As a particular case, a closed form solution is provided for homogeneous layers.

It is worth mentioning that analogous problems of existence and uniqueness have been investigated in electromagnetism (cf. [1-4]). The basic point in such investigations is the expression of the pertinent equations as a first-order quasi-linear hyperbolic system while the boundary conditions state that a linear combination of the unknown functions, at appropriate planes, is a given function of time. As it stands, this scheme does not apply in mechanics as also Bassanini remarks that in seismology it is not easy to do with the method unless proper changes are made [1]. As a further difference, electromagnetism is ordinarily applied to homogeneous layers while this paper involves an inhomogeneous layer.

2. Basic framework.

Consider a layer of thickness $L$ sandwiched between two half spaces. Let $z$ be the Cartesian coordinate such that $z \in (0, L)$ is the layer and $z < 0$ and $z > L$ are the half spaces. The layer is occupied by an inhomogeneous, isotropic elastic material. The displacement $u$ is taken to depend on the position vector $x$ only through $z$. Longitudinal (along the direction of propagation) and transverse components of $u$ satisfy decoupled differential equations of the form

\begin{equation}
\partial_z (\chi(z) \partial_z u) + f(z, t) = g(z) \partial^2_t u,
\end{equation}

where $u(z, t)$ is the unknown function on $(-\infty, 0) \cup (0, L) \cup (L, \infty)$.
Existence and uniqueness for wave propagation etc.

$\times \mathbb{R}^+$ while $f(z, t), \chi(z)$ and $\varphi(z)$ are known functions. In a standard notation, $\chi = 2\mu + \lambda$ or $\chi = \mu$ when longitudinal or transverse polarizations are considered. Also, $\varphi$ is the mass density and $f$ is the body force per unit volume. It is natural to assume that $\chi > 0$ and $\varphi > 0$ on $(-\infty, 0) \cup \cup (0, L) \cup (L, \infty).$ Since $u$ and $\chi \partial_z u$ denote the displacement and the traction, we assume that $u(\cdot, t)$ and $\chi(\cdot) \partial_z u(\cdot, t)$ are continuous everywhere.

Both $\chi$ and $\varphi$ are constant in the half spaces say $\chi(z) = \chi(0_-), \varphi(z) = \varphi(0_-)$ and $\chi(z) = \chi(L_+), \varphi(z) = \varphi(L_+)$ as $z < 0$ and $z > L.$ Consequently, in the half spaces $z < 0$ and $z > L$ the differential equation (2.1) takes the form

$$c^2 \partial_z^2 u(z, t) - \partial_t^2 u(z, t) + f(z, t)/\varphi = 0$$

where $c^2 = \chi/\varphi.$ Let $c_-, c_+$ be the values of $c$ as $z < 0, z > L.$ Any solution $u$ to (2.2) can be written as

$$u(z, t) = u^f(z - ct) + u^b(z + ct),$$

the superscript $f (b)$ being a reminder that the function describes a forward (backward) propagating wave.

We have in mind the reflection-transmission process originated by a wave coming from the half space $z < 0.$ This means that the forward propagating wave in $z < 0$ is known and no wave is coming from $z > L.$ The incident wave $u^i$ is a known function of $z - c_- t,$

$$u^i(z, t) = g(z - c_- t),$$

whereas the reflected wave is an unknown function $u^r$ of $z + c_- t.$ Hence,

$$u(z, t) = g(z - c_- t) + u^r(z + c_- t), \quad z < 0.$$

It follows that

$$(\partial_t u - c_- \partial_z u)(z, t) = -2c_- g'(z - c_- t),$$

a prime denoting differentiation with respect to the argument. Letting

$$\phi(t) := \lim_{z \to 0_-} [-2c_- g'(z - c_- t)]$$

we have

$$(\partial_t u - c_- \partial_z u)(0_-, t) = \phi(t).$$
The continuity of $u$ and $\chi \partial_z u$ yields

$$(\partial_t u - c_- \partial_z u)(0_- , t) = \partial_t u(0_+ , t) - c_- \frac{\chi(0_+)}{\chi(0_-)} \partial_z u(0_+ , t).$$

Further, assume $u^i(0_-, t) = 0$ up to $t = 0$. Hence the solution $u$ in the layer is subject to the boundary condition

$$\partial_t u(0_+, t) - c_- \frac{\chi(0_+)}{\chi(0_-)} \partial_z u(0_+, t) = \phi(t), \quad \forall t > 0. \quad (2.3)$$

For $z > L$, only the transmitted wave (namely a forward propagating wave) occurs. Accordingly,

$$u(z, t) = u^t(z - c_+ t), \quad z > L, \quad t > 0.$$  

Hence we have

$$\partial_t u(z, t) + c_+ \partial_z u(z, t) = 0, \quad z > L, \quad t > 0.$$  

We make use again of the continuity condition to obtain

$$\partial_t u(L_-, t) + c_+ \frac{\chi(L_-)}{\chi(L_+)} \partial_z u(L_-, t) = 0, \quad \forall t > 0. \quad (2.4)$$

Furthermore, we assume that the initial value of $u$ is known, namely,

$$u(z, 0) = h(z), \quad z \in [0, L]. \quad (2.5)$$

The conditions (2.3) and (2.4) mean that no incoming wave occurs except the incident one. As we show in a moment, these conditions, along with (2.5), imply the uniqueness in the layer and in the half spaces.

3. Uniqueness theorems.

Preliminarily, we state and prove the uniqueness of the solution in the layer.

**Theorem 1 (Layer).** *The solution $u$ to (2.1) subject to*

$$\partial_t u(0_+, t) - a \partial_z u(0_+, t) = \phi(t), \quad \partial_t u(L_-, t) + b \partial_z u(L_-, t) = 0,$$

$$t \in \mathbb{R}^+, \quad a, b > 0, \quad u(z, 0) = h(z), \quad z \in [0, L],$$

*is unique in $C^2([0, L] \times \mathbb{R}^+)$.***
Let \( u_1, u_2 \) be two solutions and consider their difference \( v = u_1 - u_2 \). We have
\[
\partial_z (\chi(z) \partial_z v) = \varrho(z) \partial_t^2 v, \quad z \in (0, L), \quad t \in \mathbb{R}^+,
\]
\[
\partial_t v(0^+, t) - a \partial_z v(0^+, t) = 0, \quad \partial_t v(L^-, t) + b \partial_z v(L^-, t) = 0,
\]
\[
t \in \mathbb{R}^+, \quad a, b > 0, \quad \psi(z, 0) = 0, \quad z \in [0, L].
\]
Upon multiplication by \( \partial_t v \) and exchange of the order of differentiation we have
\[
(3.1) \quad 0 = \partial_t v[\partial_z (\chi(z) \partial_z v) - \varrho(z) \partial_t^2 v] =
\]
\[
= \partial_z [\chi \partial_z v \partial_t^2 v] - \partial_z \left[ \frac{1}{2} \chi (\partial_z v)^2 + \frac{1}{2} \varrho(\partial_t^2 v)^2 \right].
\]
Integration over \([0, L]\) and use of the boundary conditions yield
\[
\frac{1}{2} \partial_t \int_0^L [\chi (\partial_z v)^2 + \varrho(\partial_t^2 v)^2] \, dz =
\]
\[
= \chi(L^-) \partial_z v(L^-, t) \partial_t v(L^-, t) - \chi(0^+) \partial_z v(0^+, t) \partial_t v(0^+, t) =
\]
\[
= -b \chi(L^-) (\partial_t v(L^-, t))^2 - a \chi(0^+) (\partial_t v(0^+, t))^2 \leq 0.
\]
Letting
\[
E(t) := \frac{1}{2} \int_0^L [\chi (\partial_z v)^2 + \varrho(\partial_t^2 v)^2] \, dz,
\]
we have
\[
E(t) \geq 0, \quad E(0) = 0, \quad \partial_t E(t) \leq 0,
\]
whence it follows that \( E(t) = 0 \) for all \( t \in \mathbb{R}^+ \). The continuity of the integrand then implies that
\[
\partial_t v(z, t) = 0 \quad \forall z \in (0, L), \quad \forall t \in \mathbb{R}^+.
\]
Since \( \psi(z, 0) = 0 \) it follows that \( v = 0 \) in \([0, L] \times \mathbb{R}^+\), which implies uniqueness. \( \blacksquare \)
Given the incident wave (in \( z < 0 \)) the solution \( u \) in the layer (namely, in \((0, L) \times R^+\)) is unique. Consequently, given \( u'(z, t), z < 0, t \geq 0 \), the functions

\[
u(0_+, t), \quad u(L_-, t)\]

are known for any \( t \in R^+ \). Since \( u \) is continuous across any interface, we conclude that also \( u(0_-, t) \) and \( u(L_+, t) \) are known for any \( t \in R^+ \). The natural question now arises as to the uniqueness of \( u \) for \( z < 0 \) and \( z > L \). This question is answered as follows.

**Theorem 2 (Half spaces).** For every finite \( T > 0 \) the solution \( u \) to (2.1) subject to

\[
u(0_-, t) = \psi(t), \quad u(L_+, t) = \eta(t), \quad t \in R^+,
\]

\[
u(z, 0) = \eta(z) \forall z \in (-\infty, 0) \cup [L, \infty),
\]

\[
\partial_t u(z, t), \partial_z u(z, t) = o(|z|^{-1/2}) \forall t \in [0, T],
\]

is unique in \( C^2((-\infty, 0] \cap [L, \infty) \times [0, T]) \).

The difference \( v \) of two solutions satisfies the differential equation (2.1) and the conditions

\[
v(0_-, t) = 0, \quad v(L_+, t) = 0, \quad t \in R^+,
\]

\[
v(z, 0) = 0 \forall z \in (-\infty, 0) \cup (L_+, \infty),
\]

\[
\partial_t v(z, t), \partial_z v(z, t) = o(|z|^{-1/2}) \forall t \in [0, T].
\]

We integrate the equality (3.1) on \((-\infty, 0)\) to obtain

\[
\frac{1}{2} \int_{-\infty}^{0} \left[ \chi(\partial_z v)^2 + g(\partial_z v)^2 \right] dz =
\]

\[
= \chi(0_-) \partial_z v(0_-, t) \partial_z v(0_-, t) - \lim_{z \to -\infty} \chi(0_-) \partial_z v(z, t) \partial_z v(z, t).
\]

The integral is bounded and the limit is zero. Hence, letting

\[
E_1(t) := \int_{-\infty}^{0} \left[ \chi(\partial_z v)^2 + g(\partial_z v)^2 \right] dz
\]
we have

\[ E_1(t) \geq 0, \quad \partial_t E_1(t) = 0, \quad E_1(0) = 0, \]

whence

\[ E_1(t) = 0 \quad \forall t \in [0, T]. \]

The vanishing of \( v \) at \( t = 0 \) yields \( v(z, t) = 0 \) at any \( z \in (-\infty, 0] \) and any \( t \in [0, T] \).

A similar procedure yields the vanishing of \( v \) as \( z \in (L, \infty) \). This is tantamount to the uniqueness of \( u \) for \( z \in (-\infty, 0-] \cup [L+, \infty) \). \( \blacksquare \)

Owing to the finite wave speed in the half spaces \( z < 0 \) and \( z > L \), for \( t \in [0, T] \) the functions \( \partial_t u \) and \( \partial_z u \) have compact support. The hypothesis \( \partial_t u(z, t), \partial_z u(z, t) = o(|z|^{-1/2}) \) is then weaker than the finiteness of the wave speed.

It is worth looking also to the cases in which the half space \( z > L \) is empty or rigid. In the first case the boundary \( z = L \) is free, in the second one it is fixed. Accordingly, the boundary condition, at \( z = L \), becomes

\[ \partial_z u(L, t) = 0 \quad \text{or} \quad u(L, t) = 0 \]

for all \( t \in \mathbb{R}^+ \). The statement and proof of Theorem 1 are then modified to give the following

**Theorem 3 (Free or fixed boundary).** The solution \( u \) to (2.1) subject to

\[ \partial_t u(0+, t) - a \partial_z u(0+, t) = \phi(t), \quad a > 0, \quad \partial_z u(L-, t) = 0 \quad \text{or} \quad u(L-, t) = 0, \]

\[ t \in \mathbb{R}^+, \quad u(z, 0) = h(z), \quad z \in [0, L], \]

is unique in \( C^2([0, L] \times \mathbb{R}^+) \).

We parallel Theorem 1 except for the boundary condition at \( z = L \) which reads \( \partial_z v = 0 \) or \( v = 0 \) for every \( t \in \mathbb{R}^+ \). In both cases we find that

\[ \frac{1}{2} \partial_z \int_0^L [\phi(\partial_z v)^2 + \phi(\partial_t v)^2] \, dz = -a\phi(0+) [\partial_t v(0+, t)]^2 \leq 0. \]
Hence it follows that $v$ is identically zero, and then $u$ unique, on $[0, L] \times \mathbb{R}^+$.  

4. Uniqueness in a scattering process.

A scattering process may be viewed as a reflection-transmission originated by incident waves impinging on the layer from both sides $z < 0$ and $z > L$ (cf. [5]). Let $u^+ \downarrow$ and $u^+ \uparrow$ be the incident waves coming from $z < 0$ and $z > L$, respectively. They produce a backward-propagating wave $u^b$ in $z < 0$ and a forward-propagating wave $u^f$ as $z > L$. Now, $u^+ \downarrow$ produces a reflected wave $u^r$ and a transmitted wave $u^t \downarrow$ while $u^+ \uparrow$ produces a reflected wave $u^r \uparrow$ and a transmitted wave $u^t \uparrow$. By the linearity of the equation (2.1), together with the initial condition $u(z, 0) = f(z)$, the simultaneous occurrence of $u^+ \downarrow$ and $u^+ \uparrow$ produces $u^r \downarrow + u^r \uparrow$ at $z < 0$ and $u^t \downarrow + u^t \uparrow$ at $z > L$. Hence uniqueness associated of $u^+ \downarrow$ and $u^+ \uparrow$ when they occur separately implies uniqueness in the case of their simultaneous occurrence.

Granted uniqueness, we look for existence of the solution to a reflection-transmission problem.

5. Reflection-transmission problem.

In essence, a reflection-transmission problem consists in determining the reflected and transmitted waves in terms of the incident one. Mathematically, this is not a standard initial and/or boundary value problem. That is why the existence of the solution involves questions which are quite specific to this problem.

Given an incident wave $u^i$ in the half space $z < 0$, we have to determine the reflected (backward propagating in $z < 0$) and the transmitted (forward propagating in $z > L$) waves. A direct application of (2.1) in the time domain, for any time dependence, seems a formidable problem. We then apply Fourier analysis to work in the frequency domain. For formal simplicity we let $f = 0$.

Preliminarily, observe that in terms of $u$ and $\tau = \chi \partial_z u$ we can express (2.1) in the form

$$\partial_z u = \gamma \tau, \quad \partial_z \tau = \partial_z^2 u,$$

where $\gamma = 1/\chi$. On the assumption that $u(z, \cdot), \partial_z u(z, \cdot), \partial_z^2 u(z, \cdot)$,
\( \tau(z, \cdot) \in L^1(\mathbb{R}) \) we apply the Fourier transform, e.g.,

\[
u_{F}(z, \omega) = \int_{-\infty}^{\infty} u(z, t) \exp(-i\omega t) \, dt,
\]
to obtain

\[
(5.1) \quad \partial_z \nu_{F} = \gamma \nu_{F}, \quad \partial_t \nu_{F} = -Q \omega^2 \nu_{F}.
\]

Let \( \nu = [\nu_{F}, \tau_{F}]^T \in \mathbb{C}^2 \). Hence (5.1) is a first-order system for \( \nu(z) \) parameterized by \( \omega \). The system (5.1) can be given the form

\[
(5.2) \quad \partial_z \nu = A \nu,
\]

where

\[
A = \begin{bmatrix}
0 & \gamma \\
-Q \omega^2 & 0
\end{bmatrix}.
\]

It is convenient (cf. [6, 7]) to diagonalize the matrix \( A \). Let \( p_1, p_2 \in \mathbb{C}^2 \) be the eigenvectors associated with the eigenvalues \( i\xi_1, i\xi_2 \in \mathbb{C} \),

\[
A p_\alpha = i\xi_\alpha p_\alpha, \quad \alpha = 1, 2.
\]

We find that

\[
\xi_1 = -\xi_2 = \sqrt{\gamma/\omega}, \quad p_1 = \begin{bmatrix}
-i\sqrt{\gamma/\omega} \\
1
\end{bmatrix}, \quad p_2 = \begin{bmatrix}
i\sqrt{\gamma/\omega} \\
1
\end{bmatrix}.
\]

Since the eigenvectors are linearly independent, we can express \( \nu \) as

\[
\nu = \sum_\alpha s_\alpha p_\alpha.
\]

It is understood that \( s_1, s_2 \) depend on \( z \) and are parameterized by \( \omega \).

6. Reflected and transmitted waves.

To functions in the frequency domain we associate functions in space-time through the inverse Fourier transform, that is,

\[
u_{F}(z, \omega) \rightarrow u(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \nu_{F}(z, \omega) \exp(i\omega t) \, d\omega.
\]
Multiplication by \( \exp(i\omega t) \) allows us to regard the functions
\[
p(\omega) \exp(i\alpha(\omega) z)
\]
as forward- or backward-propagating waves according as \( \alpha < 0 \) or \( \alpha > 0 \).
With this observation, we see that \( \xi_1 \) and \( p_1 \) represent a backward-propagating wave, \( \xi_2 \) and \( p_2 \) a forward-propagating wave. For \( z > 0 \) the incident and reflected waves \( w' \), \( w'' \) occur. They are given by
\[
w'(z, \omega) = v'(\omega) p_2(\omega) \exp(-i\omega \sqrt{\gamma} z),
\]
\[
w''(z, \omega) = v''(\omega) p_1(\omega) \exp(i\omega \sqrt{\gamma} z),
\]
where \( v'(\omega) \) is known. In the half-space \( z > L \), we have the transmitted wave \( w'' \) given by
\[
w''(z, \omega) = v''(\omega) p_2(\omega) \exp(-i\omega \sqrt{\gamma}(z - L)).
\]

For \( z \in (0, L) \), the solution \( w \) can be determined by means of the propagator matrix, that is, the matrix \( \Omega \) such that
\[
w(z, \omega) = \Omega(z, \omega) w(0, \omega),
\]
where
\[
(6.1) \quad \delta_z \Omega(z, \omega) = A(z, \omega) \Omega(z, \omega), \quad \Omega(0, \omega) = 1.
\]
Once \( \Omega(z, \omega) \) is determined, since \( w \) and \( \Omega(z, \omega) \) are continuous, we can write
\[
w(L, \omega) = \Omega(L, \omega) w(0, \omega).
\]
The reflection-transmission process is then governed by the conditions
\[
w'(0, \omega) + w''(0, \omega) = w(0, \omega), \quad \Omega(L, \omega) w(0, \omega) = w'(L, \omega).
\]
Let \( \Omega \) stand for \( \Omega(L, \omega) \). We have
\[
v' p_2 + v'' p_1 = w(0, \omega), \quad \Omega w(0, \omega) = v' p_2
\]
whence
\[
(6.2) \quad v' p_2 - v'' \Omega p_1 = v' \Omega p_2.
\]
Once the solution for \( v' \) and \( v'' \) is determined, we can write the solution for \( w' \) and \( w'' \).
Since $\text{tr} A = 0$, it follows that $\det \Omega(z, \omega)$ is constant in $z$ and hence, by the initial condition, $\det \Omega(z, \omega) = 1$.

Let $q$ be perpendicular to $p$, namely

$$q_2 = \left[ \frac{1}{-i\sqrt{\gamma/\omega}} \right], \quad q_1 = \left[ \frac{1}{i\sqrt{\gamma/\omega}} \right].$$

The solution $v^r, v^t$ to (6.2) is found to be

$$(6.3) \quad \frac{v^r}{v^t} = v(\omega) := -\frac{q_2^+ \cdot \Omega p_2}{q_2^+ \cdot \Omega p_1} \omega, \quad \frac{v^t}{v^t} = \zeta(\omega) := \frac{q_1^- \cdot \Omega p_2}{q_1^- \cdot \Omega p_1} \omega.$$

This in turn gives

$$w^r(0, \omega) = v^r(\omega) p_1^- (\omega) = v^t(\omega) v(\omega) \left[ -i\sqrt{\gamma_-/\omega} \right],$$

$$w^t(0, \omega) = v^t(\omega) p_2^- (\omega) = v^t(\omega) \zeta(\omega) \left[ i\sqrt{\gamma_+/\omega} \right].$$

Since

$$u^t_\pm (0, \omega) = v^t(\omega) i\sqrt{\gamma_-/\omega},$$

we find that

$$u^r(0_-, t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} v(\omega) u^t_\pm (0_-, \omega) \exp(i\omega t) d\omega,$$

$$u^t(L_+, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta(\omega) \sqrt{\frac{\gamma_+/\omega}{\gamma_-/\omega}} u^r_\pm (0_-, \omega) \exp(i\omega t) d\omega.$$

The functions $v, \zeta$ are generally dependent on $\omega$. Letting $F$, $G$ be their inverse Fourier transforms, we can write $u^r$ and $u^t$ in the convolution forms

$$(6.4) \quad u^r(0_-, t) = -\int_{-\infty}^{\infty} F(s) u^t_\pm (0_-, t-s) ds,$$

$$(6.5) \quad u^t(L_+, t) = \sqrt{\frac{\gamma_+/\omega}{\gamma_-/\omega}} \int_{-\infty}^{\infty} G(s) u^r_\pm (0_-, t-s) ds.$$
If \( u'(0_-, t) \) is zero up to \( t = 0 \), then the integrands in (6.4), (6.5) are zero for \( s \geq t \). Also, it is natural to require that \( F \) and \( G \) be causal, namely \( F(s), G(s) = 0 \) for \( s < 0 \). Hence (6.4) and (6.5) reduce to

\[
\begin{align*}
\int_0^t F(s) u'(0_-, t - s) \, ds,
\int_0^t G(s) u'(0_-, t - s) \, ds.
\end{align*}
\]

Incidentally, causality may be satisfied automatically (cf. next section) or may be a criterion for the choice of the integration contour for the inverse Fourier transform (cf. [8]).

7. Homogeneous layer.

When \( \gamma \) and \( \varrho \) are constant - say \( \gamma = \gamma_0, \varrho = \varrho_0 \) - the system of equations (6.1) may be solved in closed form. We find that

\[
\Omega(z, \omega) = \begin{bmatrix}
\cos \left( \sqrt{\varrho_0} \gamma_0 \omega z \right) & \left( 1/\omega \right) \sqrt{\varrho_0} \gamma_0 \sin \left( \sqrt{\varrho_0} \gamma_0 \omega z \right) \\
-\omega \sqrt{\varrho_0} \gamma_0 \sin \left( \sqrt{\varrho_0} \gamma_0 \omega z \right) & \cos \left( \sqrt{\varrho_0} \gamma_0 \omega z \right)
\end{bmatrix},
\]

where \( z \in [0, L] \). By (6.3) we find that

\[
\nu(\omega) =
\]

\[
\left( \sqrt{\varrho_-} - \sqrt{\varrho_+} \right) \cos \left( \sqrt{\varrho_0} \gamma_0 \omega L \right) - i \sqrt{\varrho_0} \gamma_0 \left( 1 - \frac{\varrho_0}{\gamma_0} \sqrt{\varrho_-} \sqrt{\varrho_+} \right) \sin \left( \sqrt{\varrho_0} \gamma_0 \omega L \right)
\]

\[
\left( \sqrt{\varrho_-} + \sqrt{\varrho_+} \right) \cos \left( \sqrt{\varrho_0} \gamma_0 \omega L \right) + i \sqrt{\varrho_0} \gamma_0 \left( 1 + \frac{\varrho_0}{\gamma_0} \sqrt{\varrho_-} \sqrt{\varrho_+} \right) \sin \left( \sqrt{\varrho_0} \gamma_0 \omega L \right)
\]

and

\[
\zeta(\omega) =
\]

\[
\frac{2}{\sqrt{\varrho_-}} \left( \sqrt{\varrho_-} + \sqrt{\varrho_+} \right) \cos \left( \sqrt{\varrho_0} \gamma_0 \omega L \right) + i \sqrt{\varrho_0} \gamma_0 \left( 1 + \frac{\varrho_0}{\gamma_0} \sqrt{\varrho_-} \sqrt{\varrho_+} \right) \sin \left( \sqrt{\varrho_0} \gamma_0 \omega L \right)
\]
It is interesting to observe that, letting
\[ n(v) = \lim_{v \to 0} n(v), \quad \xi(v) = \lim_{v \to 0} \xi(v), \]
we find that \( n \) and \( \xi \) are constants, namely,
\[ n = \frac{\sqrt{\gamma_+/\rho_+} - \sqrt{\gamma_-/\rho_-}}{\sqrt{\gamma_+/\rho_+} + \sqrt{\gamma_-/\rho_-}}, \]
\[ \xi = \frac{2\sqrt{\gamma_-/\rho_-}}{\sqrt{\gamma_-/\rho_-} + \sqrt{\gamma_+/\rho_+}}. \]
Hence we have
\[ u_r(0, t) = -n u_i(0, t), \]
\[ u_t(0, t) = \frac{\sqrt{\gamma_+/\rho_+} + \sqrt{\gamma_-/\rho_-}}{2\sqrt{\gamma_-/\rho_-}} u_i(0, t). \]
Since \( 1/\sqrt{\gamma\rho} \) is the wave speed \( c \), we can write
\[ \begin{align*}
\rho &= \frac{c_+ \gamma_+ - c_- \gamma_-}{c_+ \gamma_+ + c_- \gamma_-} = \frac{q - c_- - q_+ c_+}{q - c_+ + q_+ c_+}, \\
\xi &= \frac{2\gamma_+ c_+}{\gamma_+ - c_- + \gamma_- c_+} = \frac{2q - c_-}{q - c_+ + q_+ c_+}. \end{align*} \]
The ratios (7.1) and (7.2) for the reflection and transmission coefficients are a well-known result [9] for the discontinuity between two elastic half-spaces.

8. Conclusions.

The reflection-transmission problem is considered for an inhomogeneous layer sandwiched between homogeneous half spaces. The uniqueness theorems of § 3 illustrate the key role of the boundary conditions (2.4), (2.5). Also, uniqueness in the half spaces is shown to hold, provided we preliminary prove uniqueness in the layer. Finally, the forms (6.4) and (6.5) of \( u_r(0, t) \) and \( u_t(0, t) \) show that ordinarily the amplitudes of reflected and transmitted waves are given by convolutions of the incident amplitude.
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