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ABSTRACT - We show the existence of fundamental solutions for $p$-adic pseudo-differential operators with polynomial symbols.

1. Introduction.

Let $K$ be a $p$-adic field, i.e. a finite extension of $\mathbb{Q}_p$ the field of $p$-adic numbers. Let $R_K$ be the valuation ring of $K$, $P_K$ the maximal ideal of $R_K$, and $\mathcal{K} = R_K / P_K$ the residue field of $K$. The cardinality of $\mathcal{K}$ is denoted by $q$. For $z \in K$, $v(z) \in \mathbb{Z} \cup \{ + \infty \}$ denotes the valuation of $z$, $|z|_K = q^{-v(z)}$ and $ac(z) = z\pi^{-\varepsilon(z)}$ where $\pi$ is a fixed uniformizing parameter for $R_K$. Let $\Psi$ denote an additive character of $K$ trivial on $R_K$ but not on $P_K^{-1}$.

A function $\Phi : K^n \to \mathbb{C}$ is called a Schwartz-Bruhat function if it is locally constant with compact support. We denote by $\mathcal{S}(K^n)$ the $\mathbb{C}$-vector space of Schwartz-Bruhat functions over $K^n$. The dual space $\mathcal{S}'(K^n)$ is the space of distributions over $K^n$. Let $f = f(x) \in \mathcal{S}(K^n)$, $x = (x_1, \ldots , x_n)$, be a non-zero polynomial, and $\beta$ a complex number satisfying $\text{Re}(\beta) > 0$.

If $x = (x_1, \ldots , x_n)$, $y = (y_1, \ldots , y_n) \in K^n$, we set $[x, y] := \sum_{i=1}^{n} x_i y_i$.

A $p$-adic pseudo-differential operator $f(\partial, \beta)$, with symbol $|f|_{\mathcal{K}}^\beta$, is an

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operator of the form
\[ f(\partial, \beta) : \mathcal{S}(K^n) \rightarrow \mathcal{S}(K^n) \]
\[ \Phi \rightarrow \mathcal{F}^{-1}(J)^{\gamma} \Phi(x), \]
where
\[ \mathcal{F} : \mathcal{S}(K^n) \rightarrow \mathcal{S}(K^n) \]
\[ \Phi \rightarrow \int_{K^n} \Psi([-x, y]) \Phi(x) \, dx \]
is the Fourier transform. The operator \( f(\partial, \beta) \) has self-adjoint extension with dense domain in \( L^2(K^n) \). We associate to \( f(\partial, \beta) \) the following \( p \)-adic pseudo-differential equation:
\[ f(\partial, \beta) u = g, \quad g \in \mathcal{S}(K^n). \]

A fundamental solution for (1.3) is a distribution \( E \) such that \( u = E \ast g \) is a solution.

The main result of this paper is the following.

**Theorem 1.1.** Every \( p \)-adic pseudo-differential equation \( f(\partial, \beta) u = g, \) with \( f(x) \in K[x_1, \ldots, x_n] \setminus K, \) \( g \in \mathcal{S}(K^n), \) and \( \beta \in \mathbb{C}, \) \( \text{Re}(\beta) > 0, \) has a fundamental solution \( E \in \mathcal{S}(K^n). \)

The \( p \)-adic pseudo-differential operators occur naturally in \( p \)-adic quantum field theory [11], [6]. Vladimirov showed the existence of a fundamental solution for symbols of the form \( |\xi|_k^a, \ a > 0 \) [10], [11]. In [7], [6] Kochubei showed explicitly the existence of fundamental solutions for operators with symbols of the form \( |f(\xi_1, \ldots, \xi_n)|_k^a, \ a > 0, \) where \( f(\xi_1, \ldots, \xi_n) \) is a quadratic form satisfying \( f(\xi_1, \ldots, \xi_n) \neq 0 \) if \( |\xi_1|_k + \ldots + |\xi_n|_k \neq 0. \) In [8] Khrennikov considered spaces of functions and distributions defined outside the singularities of a symbol, in this situation he showed the existence of a fundamental solution for a \( p \)-adic pseudo-differential equation with symbol \( |f|_K \neq 0. \) The main result of this paper shows the existence of fundamental solutions for operators with polynomial symbols. Our proof is based on a solution of the division problem for \( p \)-adic distributions. This problem is solved by adapting the ideas developed by Atiyah for the archimedean case [1], and Igusa’s theorem on the meromorphic continuation of local zeta functions [3], [4]. The connection between local zeta functions (also called Igusa's local ze-
ta functions) and fundamental solutions of \( p \)-adic pseudo-differential operators has been explicitly showed in particular cases by Jang and Sato \([5]\), \([9]\). In \([9]\) Sato studies the asymptotics of the Green function \( G \) of the following pseudo-differential equation

\[(f(\bar{z}, 1) + m^2) u = g, \quad m > 0.\]

The main result in \([9\, \text{theorem 2.3}]\) describes the asymptotics of \( G(x) \) when the polynomial \( f \) is a relative invariant of some prehomogeneous vector spaces (see e.g. \([3, \text{Chapter 6}]\)). The key step is to establish a connection between the Green function \( G(x) \) and the local zeta function attached to \( f \).

All the above mentioned results suggest a deep connection between Igusa’s work on local zeta functions (see e.g. \([3]\)) and \( p \)-adic pseudo-differential equations.

2. Local zeta functions and division of distributions.

The local zeta function associated to \( f \) is the distribution

\[\langle |f|_K^s, \Phi \rangle = \int_{K^n} \Phi(x) |f(x)|_K^s \, dx,\]

where \( \Phi \in S(K^n), s \in \mathbb{C}, \text{Re}(s) > 0, \) and \( dx \) is the Haar of \( K^n \) normalized so that \( \text{vol}(K^n) = 1 \). The local zeta functions were introduced by Weil \([12]\) and their basic properties for general \( f \) were first studied by Igusa \([3], [4]\). A central result in the theory of local zeta functions is the following.

**Theorem 2.1** (Igusa, \([3\, \text{Theorem 8.2.1}]\)). The distribution \( |f|_K^s \) admits a meromorphic continuation to the complex plane such that \( \langle |f|_K^s, \Phi \rangle \) is a rational function of \( q^{-s} \) for each \( \Phi \in S(K^n) \). In addition the real parts of the poles of \( |f|_K^s \) are negative rational numbers.

The archimedean counterpart of the previous theorem was obtained jointly by Bernstein and Gelfand \([2]\), independently by Atiyah \([1]\). The following lemma is a consequence of the previous theorem.

**Lemma 2.1.** Let \( f(x) \in K[x_1, \ldots, x_n] \) be a non-constant polynomial, and \( \beta \) a complex number satisfying \( \text{Re}(\beta) > 0 \). Then there exists a distribution \( T \in S'(K^n) \) satisfying \( |f|_K^s T = 1 \).
PROOF. By theorem 2.1 \[|f|_k^s\] has a meromorphic continuation to \(C\) such that \(\langle |f|_k^s, \Phi \rangle\) is a rational function of \(q^{-s}\) for each \(\Phi \in \mathcal{S}(K^n)\). Let

\begin{equation}
|f|_k^s = \sum_{m \in \mathbb{Z}} c_m (s + \beta)^m
\end{equation}

be the Laurent expansion at \(-\beta\) with \(c_m \in \mathcal{S}'(K^n)\) for all \(m\). Since the real parts of the poles of \(|f|_k^s\) are negative rational numbers by theorem 2.1, it holds that \(|f|_k^s\) is holomorphic at \(s = -\beta\). Therefore

\begin{equation}
|f|_k^s c_m = 0 \quad \text{for all} \quad m < 0 \quad \text{and}
\end{equation}

\begin{equation}
|f|_k^{s+\beta} = c_0 |f|_k^s + \sum_{m=1}^\infty c_m |f|_k^s (s + \beta)^m.
\end{equation}

By using the Lebesgue lemma and (2.3) it holds that

\begin{equation}
\lim_{s \to -\beta} \langle |f|_k^{s+\beta}, \Phi \rangle = \int_{K^n} \Phi(x) \, dx = \langle 1, \Phi \rangle
\end{equation}

\begin{equation}
= c_0 |f|_k^s.
\end{equation}

Therefore we can take \(T = c_0\). \(\blacksquare\)

If \(T \in \mathcal{S}'(K^n)\) we denote by \(\mathcal{F}T \in \mathcal{S}'(K^n)\) the Fourier transform of the distribution \(T\), i.e. \(\langle \mathcal{F}T, \Phi \rangle = \langle S, \mathcal{F}(\Phi) \rangle, \Phi \in \mathcal{S}(K^n)\).

3. Proof of the main result.

By lemma 2.1 there exists a \(T \in \mathcal{S}'(K^n)\) such that \(|f|_k^s T = 1\). We set \(E = \mathcal{F}^{-1} T \in \mathcal{S}'(K^n)\) and assert that \(E\) is a fundamental solution for (1.3). This last statement is equivalent to assert that \(\mathcal{F}(\Phi) = \langle \mathcal{F}E, \mathcal{F}(g) \rangle\) satisfies \(|f|_k^s \mathcal{F}(\Phi) = \mathcal{F}(g)\). Since \(|f|_k^s \mathcal{F}(\Phi) = |f|_k^s |\mathcal{F}(\mathcal{F}E) \mathcal{F}(g)| = |f|_k^s T \mathcal{F}(g) = \mathcal{F}(g)|\), we have that \(E\) is a fundamental solution for (1.3).

4. Operators with twisted symbols.

Let \(\chi : R_K^\times \to \mathbb{C}\) be a non-trivial multiplicative character, i.e. a homomorphism with finite image, where \(R_K^\times\) is the group of units of \(R_K\). We put formally \(\chi(0) = 0\). If \(f(x) \in K[x_1, \ldots, x_n] \setminus K\), we say that \(\chi(\text{act}(f)) |f|_k^s\), with \(\beta \in \mathbb{C}\), \(\text{Re}(\beta) > 0\), is a twisted symbol, and call the
pseudo-differential operator

\[
\Phi \rightarrow f(\xi, \beta, \chi) \Phi = \mathcal{F}^{-1}(\chi(ac(f))) \mathcal{F}(\Phi), \quad \Phi \in \mathcal{S}(K^n),
\]

a twisted operator. Since the distribution \( \chi(ac(f)) \mathcal{F}(\phi) \) satisfies all the properties stated in theorem 2.1 (cf. [3, Theorem 8.2.1]), theorem 1.1 generalizes literally to the case of twisted operators. In [6, chapter 2] Kochubei showed explicitly the existence of fundamental solutions for twisted operators in some particular cases.

REFERENCES


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