Harnack’s Inequalities for Solutions to the Mean Curvature Equation and to the Capillarity Problem.

FEI-TSEN LIANG (*)

Abstract - We impose suitable conditions to obtain Harnack inequalities for solutions to the capillarity problems in terms merely of the prescribed boundary contact angle, the prescribed mean curvature and the dimension. Moreover, for solutions to mean curvature equation in a ball $B_R(x_0)$, Harnack’s inequalities are shown to hold in $B_{jR}(x_0)$ in terms merely of the mean curvature, $\lambda$ and the dimension. Furthermore, Harnack’s inequalities for neighborhoods of the boundary points will be established. We emphasize that the constant concerned are all explicitly obtained.

Let $\Omega$ be a bounded domain in $\mathbb{R}^n, n \geq 2$. Let $H(x, u(x))$ be a given Lipschitz-continuous function in $\Omega \times \mathbb{R}$. We consider solutions to the mean curvature equation of surfaces of prescribed mean curvature

\begin{equation}
\text{div} \, Tu = nH(x, u(x)) \quad \text{in} \, \Omega,
\end{equation}

where

$$Tu = \frac{Du}{\sqrt{1 + |Du|^2}}.$$

A solution of the capillarity problem can be looked at as a solution of the

E-mail: liang@math.sinica.edu.tw
equation (0.1) subject to the «contact angle» boundary condition

\[ Tu \cdot v = \cos \theta , \]

where \( v \) is the outward pointing unit normal of \( \partial \Omega \). Thus, geometrically we are considering a function \( u \) on \( \overline{\Omega} \) whose graph has the prescribed mean curvature \( H \) and which meets the boundary cylinder in the prescribed angle \( \theta \).

One main purpose of this paper is to obtain Harnack’s inequalities for solutions to the capillarity problems in terms merely of the dimension \( n \), the boundary contact angle \( \theta \) and the mean curvature \( H \). These results are formulated as Theorems 2-3. Moreover, for solutions to the mean curvature equation (0.1) in a ball \( B_R(x) \), Harnack’s inequalities are shown to hold in \( B_{\lambda R}(x) \), \( 0 < \lambda < 1 \), in terms merely of the mean curvature \( H \), \( \lambda \) and \( n \). This is formulated as Theorem 4. Furthermore, Harnack’s inequalities for neighborhoods of the boundary points will be established and formulated as Theorems 5-6.

We recall a Harnack’s inequality due to Serrin[25] which can be stated as follows:

«Suppose \( u(x) \in C^2(\Omega) \) is a non-negative solution of (0.1) in a two-dimensional ball \( B_R(x_0) \) for \( H \equiv 1 \) and suppose that \( u(x_0) = m \). Then there exist functions \( \varphi(m) > 0 \) and \( \Phi(m; r) < \infty \) in \( r < \varphi(m) \), such that \( |u(x)| < \Phi(m; |x|) \) in \( B_R(x_0) \). There holds \( \lim_{r \to 0} \Phi(m; r) = \infty \), while \( \varphi(m) \searrow 0 \) as \( |m| \to \infty \).»

In Finn [3], a simpler proof of this result is given by employing the notion of generalized solutions, together with either constructing barriers to apply comparison principles or showing gradient estimates of a special type. This new proof yields considerably improved and qualitatively different information. Indeed, the following is obtained in [3], in which it is remarkable that the one sided bound essential for the classical Harnack’s inequality does not appear:

«There exist a universal constant \( R_0 \) and a constant \( \bar{R} \) determined entirely by \( R \) for \( R > R_0 \), such that in Serrin’s result \( \varphi(m) \geq \bar{R} \) if \( R > R_0 \). Furthermore, there exist functions \( A_0^-(R; r) \) and \( A_0^+(R; r) \) such that if \( u \in C^2(\Omega) \) is a solution to (0.1) in \( B_R(x_0) \) with \( u(x_0) = m \), then

\[ A_0^- (R; |x|) < u(x) - m - 1 + \sqrt{1 - r^2} < A_0^+ (R; |x|), \]

\[ A_0^-(R; |x|) < u(x) - m - 1 + \sqrt{1 - r^2} < A_0^+ (R; |x|), \]
where \( \lim_{R \to 1} A_n^+(R; r) = \lim_{R \to 1} A_n^-(R; r) = 0 \), for all \( r \leq 1 \) and \( A_n^+(R; r) < \infty \), for all \( 0 < r < R \).

If \( R > R_0 \), \( m > 0 \), then \( g(m) = \infty \). Furthermore, a function \( A_1(R) \) exists such that \( A_1(R) \nrightarrow 0 \) as \( R \to 1 \) and

\[
\| u - m - 1 + \sqrt{1 - r^2} < A_1(R) \).
\]

In case \( n = 2 \) and \( H \) satisfies monotonicity condition instead of being constant, Finn and Lu [6] obtained gradient estimates of a type analogous to that employed in [3], which immediately yields the following, in which the one sided bound does not appear either.

Assume \( H'(u) \geq 0, H(-\infty) \neq H(+\infty) \). Then there exist a positive constant \( q^+(u_0; R) \leq R \) and a continuous function \( A^+_\infty(u_0; R; q) \) with \( A^+_\infty(u_0; R; 0) = u_0 \) such that if \( u(x) \) satisfies (0.1) in a two-dimensional ball \( B_\infty(x_0) \) and \( u(x_0) = u_0 \), then \( u \leq A^+_\infty \) throughout \( B_{a^+}(x_0) \).

There also exist a positive function \( q^-(u_0; R) \leq R \) and a continuous function \( A^-_\infty(u_0; R; q) \) with \( A^-_\infty(u_0; R; 0) = u_0 \) such that \( u \geq A^-_\infty \) throughout \( B_{a^-}(x_0) \).

If \( H(+\infty) = +\infty \) and \( H(-\infty) = -\infty \), then the functions \( A^+_\infty - u_0 \) and \( u_0 - A^-_\infty \) do not depend on \( u_0 \), and additionally \( q^+ = q^- = R \).

Indeed, the gradient estimates resorted to in [3] and [6] take the following form and is proved in [5], [15] and [6], respectively.

Let \( R > R_0 = 0.5654062332 \ldots \) Let \( \Omega \subset \mathbb{R}^2 \) be a “moon” domain bounded by two circular arcs \( \Gamma_1 \) and \( \Gamma_2 \) of the respective radius \( R \) and \( \frac{1}{2} \) such that \( 2 |\Omega| = |\Gamma_1| - |\Gamma_2| \), where \( |\cdot| \) denotes either the Haussdorff 2-measure or 1-measure. Let \( R(R) \) be the radius of the largest disk concentric to \( B_\infty(x_0) \) such that \( B_\infty(x_0) \subset \Omega \). There exists a positive function \( A(R; \epsilon) < \infty \) such that for any solution \( u(x) \) of (0.1) in \( B_\infty(x_0) \) and any \( \epsilon < 0 \), there holds \( |\nabla u(x)| < A(R; \epsilon) \) in \( B_{R-\epsilon}(x_0) \). Furthermore, the value \( R \) cannot be improved.

Assume \( H'(u) \geq 0, H(-\infty) \neq H(+\infty) \). Let \( u(x) \) be a solution of (0.1) over a two-dimensional disk \( B_\infty(x_0) \). Then \( |\nabla u(x_0)| \) is bounded, depending only on \( R \) and on \( u(x_0) \). If \( H(-\infty) = -\infty \) and \( H(+\infty) = +\infty \), then the bound depends only on \( R \).

The remarkable feature of this type of gradient estimate is that it de-
pends on neither boundary data nor bounds of any sorts, in contrast to results in [1] [7] [8] [9] [14] [17] [26] [27], for example. Progress aimed at obtaining gradient estimates of this type are made in [16] [17] [18].

The above-mentioned Harnack's inequalities and gradient estimates, however, has the disadvantage that, while it guarantees the existence of upper or lower bounds, the explicit value of the bounds are not known. In this paper, Harnack's inequalities are shown to hold under some imposed condition, in particular, under the one-sided bounded condition; however, the constant concerned are all explicitly obtained.

In [28], Harnack’s inequalities are obtained for nonnegative bounded solutions $u \in W^{2,\infty}(\Omega)$ of (0.1) in which $H(x, u)$ satisfies structure conditions of different feature than those required in this paper (cf. (0.12) below). The Harnack’s inequalities in [28] takes the form $\sup_{B_R} u \leq C \inf_{B_R} u$ for any ball $B_R \subset \Omega$ and $0 < \sigma < 1$, in which the constant $C$ can be explicitly calculated in terms of $n, \sigma, R$, the upper bound of $u$ in $B_R$ and the quantities involved in the structure conditions.

0. Introduction.

0.1. Preliminary Harnack’s inequalities.

Of basic importance is the following Preliminary Harnack’s Inequality, which estimate the growth of a solution in a small ball $B$ in terms of the ratio of the measure of level sets inside this ball $B$ to the whole ball $B$. Namely,

**Proposition 1 (the Preliminary Harnack’s Inequality).** Let $u$ be a $C^2(\Omega)$ function over a domain $\Omega \subset \mathbb{R}^n$, with subgraph

$$U = \{(x, t) \in \Omega \times \mathbb{R}, \ t < u(x)\}.$$

For points $\bar{z} = (\bar{x}, \bar{t}) \in \Omega \times \mathbb{R}$ and for $r > 0$, we set

$$U_r(\bar{z}) = C_r(\bar{z}) \cap U, \ \text{and} \ U_r'(\bar{z}) = C_r(\bar{z}) U,$$

with

$$C_r(\bar{z}) = \{(x, t) : |x - \bar{x}| < r, \ |t - \bar{t}| < r\}.$$

Suppose there exist positive constants $\alpha_n$ and $R^*$ depending only on $n$,
Harnack’s inequalities for solutions etc. 61

\[ \inf_{\Omega \times R} H \text{ and } \sup_{\Omega \times R} H \text{ such that} \]

(0.3) \[ |U_r(\zeta)| \geq \alpha_a r^{n+1} \text{ for all } r \leq \min (R^*, \text{dist}(\zeta, \partial(\Omega \times R))) \]

if \[ |U_r(\zeta)| > 0 \text{ for all } r > 0, \]

and

(0.4) \[ |U'_r(\zeta)| \geq \alpha_a r^{n+1} \text{ for all } r \leq \min (R^*, \text{dist}(\zeta, \partial(\Omega \times R))) \]

if \[ |U'_r(\zeta)| > 0 \text{ for all } r > 0, \]

Let us set, for \( \beta > 0 \),

\[ D_\beta = \{ x : x \in \Omega, |Du| \geq \beta \}, \quad \overline{R}^* = \max \left( R - 2R^*, \frac{3}{4} R \right), \]

and

\[ A^* = A_{K, \beta}^* = \min (1, \beta) \max \left( \frac{R^*}{R}, \frac{1}{8} \right). \]

If the ball \( B_R(x_0) \) has the radius \( R \leq \min (\overline{R}^*, \text{dist}(x_0, \partial \Omega)) \), then there exist two positive constants \( \xi_{n, a}, \alpha_{a, \beta} \) determined completely by \( \alpha_a, \beta \) and \( n \) such that, for any \( x_1 \in B_{\overline{R}^*}(x_0) \) with \( B_{2A^*}(x_1) \subset D_\beta \), we have

(0.5) \[ u(x_0) - m_{\Omega_0} \leq \xi_{n, a} \omega_a |u(x_1) - m_{\Omega_0}| + (2 + A_{K, \beta}^* \xi_{n, a}) \omega_a R \xi_{a, a, \beta} R^{1-a} \int_{B_R(x_0)} |Du| \, dx, \]

and

(0.6) \[ M_{\Omega_0} - u(x_0) \leq \xi_{n, a} \omega_a (M_{\Omega_0} - u(x_1)) + (2 + A_{K, \beta}^* \xi_{n, a}) \omega_a R \xi_{a, a, \beta} R^{1-a} \int_{B_R(x_0)} |Du| \, dx, \]

where we set \( M_{\Omega_0} = \sup_{\Omega_0} u \) and \( m_{\Omega_0} = \inf_{\Omega_0} u \), for any domain \( \Omega_0 \) such that \( B_R(x_0) \subseteq \Omega_0 \subseteq \Omega \). In fact, we are allowed to take

(0.7) \[ \xi_{n, a} = \frac{2^{a+2}}{a_a}, \]
and

\[(0.8)\quad C_{\alpha, \beta} = 2^{\frac{n+1}{2}} A_{\beta, \beta} \left( \frac{\omega_n}{\alpha_n} \right)^{\frac{1}{2}}.\]

In Theorem 1 and throughout this paper, we denote by $| \cdot |$ either the Hausdorff $(n+1)$-measure or the Hausdorff $n$-measure and denote by $B_s(x), s > 0, x \in \Omega$, a ball centered at $x$ and of radius $s$.

In Section 1, this Preliminary Harnack's Inequality will be proved by adapting the reasoning on pages 312-313 of Giusti [12], together with an application of the following modified version of Poincaré inequality.

**Proposition 2** (a modified version of Poincaré inequality). Suppose $w \in W^{1, p}(\Omega)$ for some $p \geq 1$ and convex $\Omega$, with

\[| \{ x \in \Omega, w(x) \leq 0 \} | \geq \alpha_1 | \Omega |.\]

If $p > 1$, then we have

\[\|w\|_p \leq C_{\alpha_1} \|Dw\|_p,\]

with

\[C_{\alpha_1} = (1 - (1 - \alpha_1)^{\frac{1}{p-1}})^{-1} \left( \frac{\omega_n}{|\Omega|} \right)^{\frac{1}{p}} (\text{diam } \Omega)^{\frac{1}{p}}.\]

If $p = 1$ and if we have, in addition,

\[| \{ x \in \Omega, w(x) \leq 0 \} | \geq \alpha_2 | \Omega |,\]

then

\[\|w\|_1 \leq C_{\alpha_1, \alpha_2} \|Dw\|_1,\]

with

\[(0.9)\quad C_{\alpha_1, \alpha_2} = \max (\alpha_1, \alpha_2) \left( \frac{1}{\alpha_1} \right)^{\frac{1}{p}} \left( \frac{1}{\alpha_2} \right)^{\frac{1}{p}} \left( \frac{\omega_n}{|\Omega|} \right)^{\frac{1}{p}} (\text{diam } \Omega)^{\frac{1}{p}}.\]

A proof of this inequality is given in [18]. We remark that inequalities of this type are indicated to hold, for example, in [20] and [29] for a class of domains with much less restrictions than that of convexity imposed here. However, in results of [20] and [29], the constants $C_{\alpha_1}$ and $C_{\alpha_1, \alpha_2}$ are not given explicitly.
For sufficiently small $r$, the number $\alpha_*$ in Proposition 1 can be estimated in terms of the mean curvature $H$ and $n$. In Appendix, we will resort to the estimates obtained in Giusti [12] for generalized solutions of the equation (0.1), taking advantage of the fact that generalized solutions for the equation (0.1) are allowed to take infinite values $+\infty$ and/or $-\infty$ in subdomains of $\Omega$ of positive $n$-Hausdorff measure. We shall obtain

**Proposition 3** (estimates for the number $\alpha_*$ in (0.3) and (0.4)). Let $u$ be a generalized solution to (0.1) in $\Omega$ with the subgraph $U$. Let $U_1(\bar{z})$, $U_2(\bar{z})$ be as in Theorem 1. If

\[ |U_1(\bar{z})| > 0 \quad \text{and} \quad |U_2(\bar{z})| > 0 \quad \text{for all} \quad r > 0, \]

then, setting

\[ \alpha_* = \frac{1}{4(n + 1) k_{(n + 1)}}, \]

with $k_{(m)}$ being the isoperimetric constant in $\mathbb{R}^m$, $m \geq 1$, and setting

\[ R_*^* = \begin{cases} \left( \frac{1}{2k_{(m)} \omega_n \left| \inf_{\Omega \times \mathbb{R}} H(x, t) \right|} \right)^{\frac{1}{m}}, & \text{if} \quad \inf_{\Omega \times \mathbb{R}} H(x, t) < 0, \\ \infty, & \text{if} \quad \inf_{\Omega \times \mathbb{R}} H(x, t) \geq 0, \end{cases} \]

\[ R_*^+ = \begin{cases} \left( \frac{1}{2k_{(m)} \omega_n \left| \sup_{\Omega \times \mathbb{R}} H(x, t) \right|} \right)^{\frac{1}{m}}, & \text{if} \quad \sup_{\Omega \times \mathbb{R}} H(x, t) \leq 0, \\ \infty, & \text{if} \quad \inf_{\Omega \times \mathbb{R}} H(x, t) > 0, \end{cases} \]

we have

\[ |U_1(\bar{z})| \geq \alpha_* r_+^{n+1} \quad \text{for all} \quad r \leq \min (R_*^*, \text{dist}(\bar{z}, \partial (\Omega \times \mathbb{R}))) \]

and

\[ |U_2(\bar{z})| \geq \alpha_* r_+^{n+1} \quad \text{for all} \quad r \leq \min (R_*^+, \text{dist}(\bar{z}, \partial (\Omega \times \mathbb{R}))). \]

Inserting the value of the number $\alpha_*$ in (0.10) into Proposition 1, we obtain
THEOREM 1 (the Preliminary Harnack’s Inequality*). Let \( u \in C^2(\Omega) \) be a solution of (0.1) in \( \Omega \). Let \( B_R(x_0), B_{2R}(x_0), A^\#_R, B^\#_R \) be as in Theorem 1. If \( x_1 \in B^\#_R(x_0) \) with \( B_{2^\#_R R}(x_1) \subset D^\#_R \), then (0.7) and (0.8) hold with \( \alpha_* \) in (0.10), \( R^* = \min(R^*, R^\#) \) and
\[
\bar{\xi}_{n, \alpha_*} = 2^{n+4}(n+1)k(n+1).
\]

0.2. Harnack’s inequalities for solutions to the capillarity problem.

Assume that \( u \in C^2(\Omega) \cap C^{0,1}(\Omega) \) is a solution to the capillarity problem (0.1) and (0.2). Furthermore, suppose that \( \partial\Omega \) is of class \( C^2 \) and that the functions
\[
H \in C^{0,1}(\Omega \times \mathbb{R}) \quad \text{and} \quad \cos \theta \in C^{0,1}(\partial\Omega)
\]
satisfy the conditions
\[
0 \leq \bar{\gamma}, \quad \text{and} \quad \inf_{x \in \partial\Omega} H \geq 0,
\]
for some positive constant \( \bar{\gamma}, 0 \leq \bar{\gamma} < 1 \).

First of all, let us extend \( \cos \theta \) and \( \nu \) into the whole domain \( \Omega \) such that \( \cos \theta \) belonging to \( C^{0,1}(\overline{\Omega}) \) still satisfies (0.12) and such that the vector field \( \nu \) is uniformly Lipschitz continuous in \( \Omega \) and bounded in absolute value by the number 1. The extensions are possible in view of the smoothness of \( \partial\Omega \).

An integration of (0.1) and (0.2) yields
\[
\int_\Omega T u \cdot D\eta \, dx + n \int_\Omega H \eta \, dx - \int_{\partial\Omega} \cos \theta \eta \, d\mathcal{H} - 1 = 0,
\]
for all \( \eta \in C^1(\overline{\Omega}) \). Henceforth, we may say that \( u \in C^1(\overline{\Omega}) \cap W^{1,1}(\Omega) \) is a solution of (0.1) and (0.2) in the weak sense in \( \Omega \) if it satisfies (0.1) in \( \Omega \) and (0.13) for all \( \eta \in C^1(\overline{\Omega}) \).

To handle the third term on the left hand side of (0.13), we recall the following result in Lemma 1.1 of Giusti [12] and its proof.

LEMMA 1 (Giusti [12]). Let \( \partial\Omega \) be of class \( C^2 \) and
\[
d(x) = \text{dist}(x, \partial\Omega),
\]
for \( x \in \Omega \). For \( \varepsilon > 0 \) which is so small that the function \( d(x) \) is of class \( C^2 \) in
\[
\Sigma_\varepsilon = \{ x \in \Omega : \text{dist}(x, \partial\Omega) \leq \varepsilon \},
\]
there exists a constant $C_{\epsilon, \Omega}$ determined completely by $\epsilon$ and $\partial \Omega$ such that

\[
\int_{\Omega} |w| dH_{n-1} \leq \int_{\Sigma_{\epsilon}} |Dw| + C_{\epsilon, \Omega} \int_{\Sigma_{\epsilon}} |w| d\Sigma,
\]

for every $w \in BV(\Omega)$. In fact, if we let $\eta_{\epsilon}$ be a $C^{\infty}$ function with

\[
\begin{align*}
0 & \leq \eta_{\epsilon} \leq 1, \\
\eta_{\epsilon} & = 1 \quad \text{on } \partial \Omega, \\
\eta_{\epsilon} & = 0 \quad \text{in } \Omega \setminus \Sigma_{\epsilon},
\end{align*}
\]

then we can take

\[
C_{\epsilon, \Omega} = \sup_{\Omega} |\text{div}(\eta_{\epsilon} Dd)|.
\]

Indeed, the following results are well known (cf. e.g. pages 354-357 of [10], pages 420-422 of [24]).

**LEMMA 2.** Let $\partial \Omega$ be of class $C^2$ whose principal curvatures are bounded in absolute value by $K_{\partial \Omega}$. Then $d(x) = \text{dist}(x, \partial \Omega)$ is of class $C^2$ in $\Sigma_{\epsilon}$, for $\epsilon \leq \frac{1}{3K_{\partial \Omega}}$, where $\Sigma_{\epsilon}$ is given in (0.13).

Furthermore, for points $\bar{x}$ in $\Sigma_{\epsilon}$, $\epsilon \leq \frac{1}{3K_{\partial \Omega}}$, define $\bar{y} = \bar{y}(\bar{x})$ to be the (unique) points on $\partial \Omega$ nearest to $\bar{x}$. Consider the special coordinate frame in which the $x_n$-axis is oriented along the inward normal to $\partial \Omega$ at $\bar{y}$ and the coordinates $x_1, \ldots, x_{n-1}$ lie along the principal directions of $\partial \Omega$ at the point $\bar{y}$. In this special coordinates, we have at $\bar{x}$

\[
Dd = (0, \ldots, 0, 1)
\]

and

\[
D^2d = \text{diagonal} \left[ \frac{k_1}{1-k_1d}, \ldots, \frac{k_{n-1}}{1-k_{n-1}d}, 0 \right],
\]

where $k_1, \ldots, k_{n-1}$ are the principal curvatures of $\partial \Omega$ at $\bar{y}$.

Inserting (0.18) and (0.19) into (0.17), we obtain the following

**LEMMA 3.** Let $\partial \Omega$ be of class $C^2$ whose principal curvatures are bounded in absolute value by $K_{\partial \Omega}$. Then, for $\epsilon \leq \frac{1}{2K_{\partial \Omega}}$ and for each $\delta$,
\[ 0 < \delta \leq 1, \text{ we can take in } (0.13) \]

\[ C_{\epsilon, \Omega} \leq |D\eta| + 2(n-1) \mathcal{H}_1 \]

\[ \leq \left( \frac{1 + \delta}{\epsilon} \right) + 2(n-1) \mathcal{H}_1. \]

Combining (0.12) and (0.15) with \( C_{\epsilon, \Omega} \) being as in (0.20), we obtain, for \( \epsilon \leq \frac{1}{2 \mathcal{H}_1} \),

\[ \int_{\Omega} \cos \theta \eta \, d\mathcal{H}_{n-1} \leq \gamma \int_{\Omega} |D\eta| + \gamma C_{\epsilon, \Omega} \int_{\Omega} |\eta| \, dx \leq \]

\[ \gamma \int_{\Omega} |D\eta| + \gamma \left( \frac{1 + \delta}{\epsilon} + 2(n-1) \mathcal{H}_1 \right) \int_{\Omega} |\eta| \, dx, \]

for all \( \eta \in C^1(\Omega) \).

In Section 2.1, we shall obtain Harnack’s inequality in the following formulation by inserting (0.21) into (0.13) either with \( \eta = (u - m_{\Omega_0}) \) or with \( \eta = (M_{\Omega_0} - u) \), and subsequently inserting what results in into (0.5) and (0.6).

**Theorem 2** (the first Harnack’s inequality). Let \( u \in C^2(\Omega) \cap W^{1,1}(\Omega) \) be a solution to (0.1) and (0.2) in the weak sense in \( \Omega \). Suppose that \( \partial \Omega \in C^2 \) such that (0.12) holds in which \( \gamma \) satisfies

\[ 2 \gamma < \left( \frac{\inf_{x \in \partial \Omega(x_0)} H}{\mathcal{H}_1} \right). \]

Let us set

\[ c_1 = (2 + \xi_{\alpha, \alpha} A_{\beta, \beta}^*) \frac{\omega_\alpha}{1 - \gamma} R + \xi_{\alpha, \alpha} C_{\alpha, \beta} \frac{|\Omega|}{(1 - \gamma) R^{n-1}} \]

and

\[ c_2 = (2 + \xi_{\alpha, \alpha} A_{\beta, \beta}^*) \omega_\alpha R + \xi_{\alpha, \alpha} C_{\alpha, \beta} \frac{|\Omega|}{R^{n-1}}. \]

Then, for any \( x_1 \in B_{R^*}(x_0) \) with \( B_{2R^*}(x_1) \subset D_{\beta} \), \( R^* \) and \( A^* \) being given
Harnack’s inequalities for solutions etc. 67

in Proposition 1), we have

\[ u(x_0) - m_{\Omega_1} \leq \frac{\xi_{n, \alpha, \omega_u}}{1 - \gamma} (u(x_1) - m_{\Omega_0}) + \mathcal{C}_1. \]  

and

\[ M_{\Omega_0} - u(x_0) \leq \frac{\xi_{n, \alpha, \omega_u}}{1 - \gamma} (M_{\Omega_0} - u(x_1)) + \mathcal{C}_1, \]

where we set \( m_{\Omega_0} = \inf_{\Omega_0} u \) and \( M_{\Omega_0} = \sup_{\Omega_0} u \), for any domain \( \Omega_0 \) such that \( B_R(x_0) \subseteq \Omega_0 \subseteq \Omega \). Furthermore, for any \( x_1 \in B_{\bar{R}_0}(x_0) \) with \( B_{2A^* R}(x_1) \subset D_\beta \),

\[ \bar{R}_0 = \max \left( \frac{R}{2} - 2R^*, \frac{3}{8} R \right), \]

we have

\[ u(x_0) - m_{\Omega_1} \leq \xi_{n, \alpha, \omega_u} (u(x_1) - m_{\Omega_0}) + \mathcal{C}_2, \]

\[ M_{\Omega_0} - u(x_0) \leq \xi_{n, \alpha, \omega_u} (M_{\Omega_0} - u(x_1)) + \mathcal{C}_2, \]

and for any \( x_1, x_2 \in B_{\bar{R}_0}(x_0) \) with \( B_{2A^* R}(x_1) \subset D_\beta \), \( \bar{R}_1 = \max \left( \frac{R}{4} - 2R^*, \frac{3}{16} R \right) \), we have

\[ u(x_1) - m_{\Omega_0} \leq \xi_{n, \alpha, \omega_u} (u(x_2) - m_{\Omega_0}) + \\
+ (2 + \xi_{n, \alpha, \omega_u} A_{\bar{R}, \beta}^\#) \omega_u \text{ dist} (x_1, x_2) + \xi_{n, \alpha, \omega_u} C_{\alpha, \beta} R_0 \text{ dist} (x_1, x_2), \]

\[ M_{\Omega_0} - u(x_1) \leq \xi_{n, \alpha, \omega_u} (M_{\Omega_0} - u(x_2)) + \\
+ (2 + \xi_{n, \alpha, \omega_u} A_{\bar{R}, \beta}^\#) \omega_u \text{ dist} (x_1, x_2) + \xi_{n, \alpha, \omega_u} C_{\alpha, \beta} R_0 \text{ dist} (x_1, x_2). \]

In the special case where \( \Omega \) is the ball \( B_R(x_0) \), we have

\[ k_i = \frac{1}{R}, \quad i = 1, \ldots, n - 1, \]  

and from (0.12) we obtain immediately

\[ \left| \int_{\partial B_R(x_0)} \cos \theta \eta d\mathcal{H}^{n-1}_{\eta} \right| \leq R^n \omega_u R^{n-1} \sup_{B_R} |\eta|. \]

In Sections 2.2 and 2.3, the inequality (0.30) is inserted into (0.5) and (0.6) to obtain the following.

**Corollary 1** (the second Harnack’s inequality). Let \( u \in C^2(\Omega) \cap W^{1,1}(\Omega) \) be a solution to (0.1) and (0.2) in the weak sense in \( B_R(x_0) \).
Suppose that (0.12) holds in which $\tilde{\gamma}$ satisfies

$$2 \tilde{\gamma} < R \inf_{x \in B_R(x_0)} H.$$  
(0.22*)

Then, for any $x_1 \in B_R(x_0)$ with $B_{2A^*R}(x_1) \subset D$, if we set

$$M_R = \sup_{B_R(x_0)} u \quad \text{and} \quad m_R = \inf_{B_R(x_0)} u,$$

then there hold either

$$u(x_0) - m_R \leq \xi_{a_1, a_2} \omega_n (u(x_1) - m_R) + c_{2, R},$$  
(0.31)

or

$$M_R - u(x_0) \leq \xi_{a_1, a_2} \omega_n (M_R - u(x_1)) + c_{2, R},$$  
(0.32)

where

$$c_{2, R} = (2 + \xi_{a_1, a_2} A^e_{R, \beta}) \omega_n R + \xi_{a_1, a_2} C_{a_1, a_2} \omega_n R.$$  
(0.33)

**Corollary 2** (the third Harnack’s inequality). Let $u \in C^2(\Omega) \cap W^{1,1}(\Omega)$ be a solution to (0.1) and (0.2) in the weak sense in $B_R(x_0)$. Suppose that (0.12) holds. Furthermore, suppose that $\tilde{\gamma} R^{n-1}$ is so small that, for some $0 < r \leq 1/2$,

$$n \omega_n \tilde{\gamma} R^{n-1} \leq \frac{r}{\xi_{a_1, a_2} C_{a_1, a_2}}.$$  
(0.34)

Then, for any $x_1 \in B_R(x_0)$ with $B_{2A^*R}(x_1) \subset D$, there holds either

$$u(x_0) - m_R \leq \frac{1}{(1 - 2r)} \xi_{a_1, a_2} \omega_n (u(x_1) - m_R) + \frac{1}{(1 - 2r)} c_{2, R},$$  
(0.35)

or

$$M_R - u(x_0) \leq \frac{1}{(1 - 2r)} \xi_{a_1, a_2} \omega_n (M_R - u(x_1)) + \frac{1}{(1 - 2r)} c_{2, R},$$  
(0.36)

where $M_R, m_R$ are given in Theorem 2 and $c_{2, R}$ is given in (0.33).

We note that the mean curvature $H$ is not involved in the condition (0.34).
0.3. Harnack’s inequalities for solutions to the mean curvature equation.

For solutions to the mean curvature equation \( \lambda \) to the mean curvature equation \( \lambda \) in \( \Omega \), we shall establish in Section 3 Harnack’s inequalities in \( \Omega \) in the following formulation in which the constants are completely determined by \( \lambda \), \( H \) and \( n \). We emphasize that no boundary condition is involved in these Harnack’s inequalities.

**Theorem 3 (the fourth Harnack’s inequality).** Let \( u \in C^2(\Omega) \) be a solution to \( (0.1) \) in \( \Omega \). Suppose that \( (\inf_{x \leq \Omega} H) \geq 0 \) for any \( \lambda \), \( 0 < \lambda < 1 \), and for any point \( x_1 \in \Omega \), \( R^* = \max \left( \lambda R - 2R^*, \frac{3IR}{4} \right) \), with \( B_{2R^*}(x_1) \subset D_R \), we have, either

\[
(0.37) \quad u(x_0) - m_R \leq \xi_{n,a,C} (1 + 2C^*_{\lambda} C_{\lambda} \omega_\lambda)(u(x_1) - m_R) + \xi_{2,R},
\]

or

\[
(0.38) \quad M_R - u(x_0) \leq \xi_{n,a,C} (1 + 2C^*_{\lambda} C_{\lambda} \omega_\lambda)(M_R - u(x_1)) + \xi_{2,R},
\]

where we set

\[
(0.39) \quad C_{\lambda} = \lambda^{-1} + \lambda^{-2} + \cdots + \lambda + 1 = \frac{1 - \lambda^n}{1 - \lambda},
\]

and

\[
(0.40) \quad C^*_{\lambda} = 1 - (\inf_{x \in \Omega} H) R \left( \frac{n}{n + 1} \right) (1 - \lambda) \left( 1 + \frac{\lambda^n}{C_{\lambda}} \right) - \frac{n(\inf_{x \in \Omega} H) R^{\frac{n+1}{C_{\lambda}}}}{C_{\lambda}}.
\]

If there holds, for some \( \tau \), \( 0 < \tau < \frac{1}{2} \),

\[
(0.41) \quad C_{\lambda} C^*_{\lambda} \omega_\lambda R^{n-1} < \frac{\tau}{\xi_{n,a,C} \lambda_{\Omega}},
\]

then we have either

\[
(0.42) \quad u(x_0) - m_R \leq \frac{1}{(1 - 2\tau)} \xi_{n,a,C} (u(x_1) - m_R) + \frac{1}{(1 - 2\tau)} \xi_{2,R},
\]
or

\[
(0.43) \quad M_R - u(x_0) \leq \frac{1}{(1 - 2r)} \xi_{u, a_*}(M_R - u(x_1)) + \frac{1}{(1 - 2r)} c_{1, R}.
\]

0.4. Boundary Harnack’s inequality.

Appealing to the following results in [12], Harnack’s inequalities for neighborhoods of boundary points can be established by the reasoning in Section 2 and Section 3.1 without essential change. Thus, we formulate the first four Harnack’s inequalities without giving a proof. In Section 3.2, we shall briefly indicate the reasoning leading to the fourth boundary Harnack’s inequality. A proof of Proposition 4 will be given in Appendix.

**Proposition 4.** Let \( u \) be a function in \( C^2(\Omega) \cap W^{1,1}(\Omega) \) in a domain \( \Omega \subset \mathbb{R}^n \) with the subgraph \( U \). Let \( U_r(z), U'_r(z) \) be as in Theorem 1. Suppose the first inequality in (0.12) holds and suppose that \( \overline{\Omega} \) is of class \( C^2 \) with \( C_e, \overline{\Omega} \) given in (0.17). If

\[
|U_r(z)| > 0 \quad \text{and} \quad |U'_r(z)| > 0 \quad \text{for all} \quad r > 0,
\]

then there exist positive constants \( R_{***}, R'_{**} \) and \( \alpha_{**} \) determined completely by \( n, \inf_{\Omega \times \mathbb{R}} H, \sup_{\Omega \times \mathbb{R}} \gamma \) and \( C_{e, \Omega} \) such that

\[
|U_r(z)| > \alpha_{**} r^{n+1} \quad \text{for every} \quad r \leq R_{**},
\]

and

\[
|U'_r(z)| > \alpha_{**} r^{n+1} \quad \text{for every} \quad r \leq R'_{**}.
\]

In particular, we can take

\[
(0.44) \quad \alpha_{**} = \frac{1 - \gamma}{16(n + 1) k_{(n + 1)}},
\]

\[
(0.45) \quad R_{**} = \begin{cases} 
\frac{C_{\gamma}}{C_{e, \Omega} k_{(n + 1)}}, & \text{if} \quad \inf_{\Omega \times \mathbb{R}} H(x, t) \geq 0, \\
\min \left( \frac{C_{\gamma}}{C_{e, \Omega} k_{(n + 1)}(2 \omega_{\omega})^{k_{(n + 1)}}} R'_{**} \right), & \text{if} \quad \inf_{\Omega \times \mathbb{R}} H(x, t) < 0,
\end{cases}
\]
and

\[
R_+^{**} = \left\{ \begin{array}{ll}
\frac{C_{\gamma}}{C_{s, \Omega} k_{(n+1)}} , & \text{if } \sup_{B \times R} H(x, t) \leq 0 , \\
\min \left( \frac{C_{\gamma}^-}{C_{s, \Omega} (k_{(n+1)})(2 \omega_{a})^{1(n+1)}}, \frac{\bar{R}_+^{**}}{R_+^{**}} \right) , & \text{if } \sup_{B \times R} H(x, t) > 0 ,
\end{array} \right.
\]

in which we set

\[
C_{\gamma} = \min \left( \frac{1}{2}, \frac{1 - \tilde{\gamma}}{3 \tilde{\gamma} + 1} \right) ,
\]

with

\[
\bar{R}_+^{**} = \left( \frac{1 - \tilde{\gamma}}{4 n(k_{(n+1)} \omega_{a}| \inf_{B \times R} H|)} \right)^{\gamma + 1} ,
\]

\[
\bar{R}_-^{**} = \left( \frac{1 - \tilde{\gamma}}{4 n(k_{(n+1)} \omega_{a}| \sup_{B \times R} H|)} \right)^{\gamma + 1} ,
\]

and

\[
C_{\gamma}^- = \min \left( \frac{1}{2}, \frac{1 - \tilde{\gamma} - 2 n(k_{(n+1)}) | \inf_{B \times R} H| \omega_{a}(\bar{R}_-^{**})^n}{3 \tilde{\gamma} + 1} \right) ,
\]

\[
C_{\gamma}^+ = \min \left( \frac{1}{2}, \frac{1 - \tilde{\gamma} - 2 n(k_{(n+1)}) | \sup_{B \times R} H| \omega_{a}(\bar{R}_+^{**})^n}{3 \tilde{\gamma} + 1} \right) .
\]

**Theorem 4** (the preliminary boundary Harnack’s inequality). Let \( u \in C^2(\Omega) \cap W^{1,1}(\Omega) \) be a solution to (0.1) and (0.2) in the weak sense in \( \Omega \).

Let us set \( a_{**} \) as in (0.44), \( R^{**} = \min(R_+^{**}, R_-^{**}) \), with \( R^{**} \) and \( R_+^{**} \) given as in (0.45) and (0.46) and set

\[
\bar{R}^{**} = \max \left( R - 2 R^{**}, \frac{3}{4} R \right) ,
\]
and

$$A_{R, \beta}^{**} = A^{**} = \min(1, \beta) \max\left(\frac{R^{**}}{R}, \frac{1}{\beta}\right).$$

If the ball has the radius $R \leq R^{**}$, then there exist two positive constants $\xi_{n, a**}$ and $C_{a**}$ determined completely by $a_{**}, \beta, \gamma$ and $n$ such that, for any $x_1 \in B_{R^{**}}(x_0) \cap \Omega$ with $B_{2A^{**}}(x_1) \cap \Omega \subset D_{\beta}$, we have

$$u(x_0) - m_{\Omega_0} \leq \xi_{n, a**} \omega_n (u(x_1) - m_{\Omega_0}) + (2 + C_{a**}) \xi_{n, a**} \omega_n R + \xi_{n, a**} C_{a**} R^{n-1} \int_{\Omega_0} |Du|dx,$$

and

$$M_{\Omega_0} - u(x_0) \leq \xi_{n, a**} \omega_n (M_{\Omega_0} - u(x_1)) + (2 + C_{a**}) \xi_{n, a**} \omega_n R + \xi_{n, a**} C_{a**} R^{n-1} \int_{\Omega_0} |Du|dx,$$

where we set $M_{\Omega_0} = \sup_{\Omega_0} u$ and $m_{\Omega_0} = \inf_{\Omega_0} u$, for any subset $\Omega_0$ of $\Omega$ with $(B_{2}(x_0) \cap \Omega) \subset \Omega_0 \subset \Omega$. In fact, we are allowed to take

$$\xi_{n, a**} = \frac{2n + 2}{a_{**}},$$

and

$$C_{a**} = 2^{n+1} \gamma \left(\frac{\omega_n}{a_{**}}\right)^{\frac{1}{n}}.$$

**Theorem 5** (the first boundary Harnack’s inequality). Let $u \in C^2(\Omega) \cap W^{1,1}(\Omega)$ be a solution to (0.1) and (0.2) in the weak sense in $\Omega$. Suppose $\Omega \in C^2$ whose principal curvatures are bounded in absolute value by $K_{\Omega}$. For $x_0 \in \partial\Omega$, suppose that the first inequality in (0.12) holds for some number $\gamma$, $0 < \gamma < 1$, in $\partial\Omega \cap B_{2}(x_0)$ and $(\inf_{x \in (B_2(x_0) \cap \Omega)} H) \geq 0$. Suppose that (0.22) holds. Suppose further that

(0.47) \quad Tu \cdot v_R \leq 0,

throughout $\partial B_{2A^{**}}(x_1) \cap \Omega \cap \Omega$, where $v_R$ is the outward unit normal with respect to $B_{2A^{**}}(x_1) \cap \Omega \cap \Omega$. Let us set $c_1$ and $c_2$ as in Theorem 2.
Then, for any \(x_1 \in B_{R_1^*}(x_0) \cap \Omega\) with \(B_{z_{2,4}^*}(x_1) \cap \Omega \subset D_{\beta}\), we have
\[
u(x_0) - m_{\Omega_0} \leq \frac{\xi_{n, a, \star} \omega_n}{1 - \gamma}(u(x_1) - m_{\Omega_1}) + c_{1,1},
\]
and
\[
M_{\Omega_0} - u(x_0) \leq \frac{\xi_{n, a, \star} \omega_n}{1 - \gamma}(M_{\Omega_1} - u(x_1)) + c_{1,1},
\]
for any subset \(\Omega_0\) of \(\Omega\) with \((B_{R'}(x_0) \cap \Omega) \subseteq \Omega_0 \subseteq \Omega\). Furthermore, for any \(x_1 \in B_{R_1^*}(x_0) \cap \Omega\), \(R_1^{**} = \max\left(\frac{R}{2} - 2R^{**}, \frac{3}{8}R\right)\), with \(B_{z_{2,4}^*}(x_1) \cap \Omega \subset D_{\beta}\), we have
\[
u(x_0) - m_{\Omega_0} \leq \frac{\xi_{n, a, \star} \omega_n}{1 - \gamma}(u(x_1) - m_{\Omega_1}) + c_{2,2},
\]
and
\[
M_{\Omega_0} - u(x_0) \leq \frac{\xi_{n, a, \star} \omega_n}{1 - \gamma}(M_{\Omega_1} - u(x_1)) + c_{2,2},
\]
and for any \(x_1, x_2 \in B_{R_1^*}(x_0) \cap \Omega\), \(R_1^{**} = \max\left(\frac{R}{4} - 2R^{**}, \frac{3}{16}R\right)\), with \(B_{z_{2,4}^*}(x_1) \cap \Omega \subset D_{\beta}\), we have
\[
u(x_1) - m_{\Omega_0} \leq \frac{\xi_{n, a, \star} \omega_n}{1 - \gamma}(u(x_2) - m_{\Omega_1}) +
\]
\[
(1 + \xi_{n, a, \star} A^{**}) \omega_n \text{dist} (x_1, x_2) + \xi_{n, a, \star} C_{a, b, \beta} \omega_n \text{dist} (x_1, x_2),
\]
\[
M_{\Omega_0} - u(x_1) \leq \frac{\xi_{n, a, \star} \omega_n}{1 - \gamma}(M_{\Omega_1} - u(x_2)) +
\]
\[
(1 + \xi_{n, a, \star} A^{**}) \omega_n \text{dist} (x_1, x_2) + \xi_{n, a, \star} C_{a, b, \beta} \omega_n \text{dist} (x_1, x_2).
\]

**Corollary 3** (the second boundary Harnack’s inequality). Let \(u \in C^2(\Omega) \cap W^{1,1}(\Omega)\) be a solution to (0.1) and (0.2) in \(B_R(x_0)\). For \(x_0 \in \partial B_R(x_0)\), suppose that the first inequality in (0.12) holds in \(\partial B_R(x_0) \cap \Omega\) and \(\inf_{x \in (B_R(x_0) \cap \partial B_R(x_0))} H \geq 0\). Suppose that (0.22) holds. Suppose further that (0.47) holds along \(\partial B_R(x_0) \cap \partial B_R(x_0)\). Then, for any \(x_1 \in B_{R_1^*}(x_0) \cap B_R(x_0)\) with \(B_{z_{2,4}^*}(x_1) \cap B_R(x_0) \subset D_{\beta}\), if we set
\[
\tilde{M}_R = \sup_{B_R(x_0) \cap B_R(x_0)} u \text{ and } \tilde{m}_R = \inf_{B_R(x_0) \cap B_R(x_0)} u,
\]
then there hold either
\[
u(x_0) - \tilde{m}_R \leq \frac{\xi_{n, a, \star} \omega_n}{1 - \gamma}(u(x_1) - \tilde{m}_R) + c_{1,1},
\]
or
\[
\bar{M}_R - u(x_0) \leq \xi_{n, a, \ast} \omega_n (\bar{M}_R - u(x_1)) + \varepsilon_{2, R}
\]
where \(\varepsilon_{2, R}\) is given in (0.33).

**Corollary 4** (the third boundary Harnack’s inequality). Let \(u \in C^2(\Omega) \cap W^{1, 1}(\Omega)\) be a solution to (0.1) and (0.2) in the weak sense in \(B_R(x_0)\). For \(x_0 \in \partial B_R(x_0)\), suppose that the first inequality in (0.12) holds in \(\partial B_R(x_0) \cap B_R(x_0)\) and \((\inf_{x \in (B_R(x_0) \cap B_R(x_0))} H) \geq 0\). Suppose (0.34) holds for some \(\tau\), \(0 < \tau \leq \frac{1}{2}\).

1. Suppose further that (0.47) holds along \(\partial B_R(x_0) \cap B_R(x_0)\). Then, for any \(x_1 \in B_{R \ast}(x_0) \cap B_R(x_0)\) with \(B_{R \ast}(x_1) \cap B_R(x_0) \subset D_R\), there holds either
   \[
   u(x_0) - \bar{m}_R \leq \frac{1}{(1 - 2\tau)} \xi_{n, a, \ast} \omega_n (u(x_1) - \bar{m}_R) + \frac{1}{(1 - 2\tau)} \varepsilon_{2, R},
   \]
or
   \[
   \bar{M}_R - u(x_0) \leq \frac{1}{(1 - 2\tau)} \xi_{n, a, \ast} \omega_n (\bar{M}_R - u(x_1)) + \frac{1}{(1 - 2\tau)} \varepsilon_{2, R}.
   \]

2. If (0.47) fails to hold throughout \(\partial B_{R \ast}(x_1) \cap \Omega\), but if
   \[
   n \omega_n \bar{R}^{n-1} + n \omega_n \bar{R}^{n-1} < \frac{\tau}{\xi_{n, a, \ast} C_{a, \ast, \beta}},
   \]
   for some \(\tau\), \(0 < \tau \leq \frac{1}{2}\), then we have
   \[
   u(x_0) - \bar{m}_R \leq \frac{1}{(1 - 2\tau)} \xi_{n, a, \ast} \omega_n (u(x_1) - \bar{m}_R) + \frac{1}{(1 - 2\tau)} \varepsilon_{2, R},
   \]
and
   \[
   \bar{M}_R - u(x_0) \leq \frac{1}{(1 - 2\tau)} \xi_{n, a, \ast} \omega_n (\bar{M}_R - u(x_1)) + \frac{1}{(1 - 2\tau)} \varepsilon_{2, R}.
   \]

**Theorem 6** (the fourth boundary Harnack’s inequality). Let \(u \in C^2(\Omega)\) be a solution to (0.1) in \(B_R(x_0)\). For \(x_0 \in \partial B_R(x_0)\), suppose that \((\inf_{x \in (B_R(x_0) \cap B_R(x_0))} H) \geq 0\). For any \(\lambda\), \(0 < \lambda < 1\), and for any point \(x_1 \in \partial B_R(x_0) \cap B_R(x_0)\), there holds either
\[
\bar{M}_R - u(x_0) \leq \frac{1}{(1 - 2\lambda)} \xi_{n, a, \ast} \omega_n (\bar{M}_R - u(x_1)) + \frac{1}{(1 - 2\lambda)} \varepsilon_{2, R},
\]
or
\[
\bar{M}_R - u(x_0) \leq \frac{1}{(1 - 2\lambda)} \xi_{n, a, \ast} \omega_n (\bar{M}_R - u(x_1)) + \frac{1}{(1 - 2\lambda)} \varepsilon_{2, R}.
\]
Harnack’s inequalities for solutions etc.

\[ \in B_{R^*}(x_0) \cap B_{R}(x_0), \quad R^{**} = \max \left( \lambda R^* - 2R^{**}, \frac{3\lambda R}{4} \right) , \] with \( B_{R^*}(x_1) \cap \cap B_{R}(x_0) \subset D_\beta ^*, \) we have, either

\begin{align}
 & u(x_0) - \bar{m}_R \leq \xi_{n,a_1^*(1 + 2C_2C_{r,\Omega} \omega_n)}(u(x_1) - \bar{m}_R) + c_{l,R} , \\
 \text{or} \quad
 & \bar{M}_R - u(x_0) \leq \xi_{n,a_1^*(1 + 2C_2C_{r,\Omega} \omega_n)}(\bar{M}_R - u(x_1)) + c_{l,R} .
\end{align}

If there holds, for some \( r, \quad r \leq \frac{1}{2}, \)

\[ C_{r,\omega_n} R^{n-1} < \frac{r}{\xi_{n,a_1^* C_{r,\Omega}}}, \]

then we have either

\begin{align}
 & u(x_0) - \bar{m}_R \leq \frac{1}{1 - 2r} \xi_{n,a_1^*(u(x_1) - \bar{m}_R)} + \frac{1}{1 - 2r} c_{l,R} , \\
 \text{or} \quad
 & \bar{M}_R - u(x_0) \leq \frac{1}{1 - 2r} \xi_{n,a_1^*(\bar{M}_R - u(x_1))} + \frac{1}{1 - 2r} c_{l,R} .
\end{align}

1. Proof of the preliminary Harnack’s inequality.

In this section, we shall prove the Preliminary Harnack’s Inequality (Proposition 1), adapting the reasoning on pages 312-313 of Giusti [12], together with an application of Proposition 2. The reasoning suggested by Giusti [12] enables us to estimate the left hand side of (0.5) and (0.6) in terms of the \( L^1 \)-norm of \( u \) and a subsequent application of Proposition 2 yields this estimate in terms of the \( L^1 \)-norm of \( |Du| \).

1.1. Suppose that \( u \in C^2(\Omega) \) satisfies (0.3) and (0.4). For any domain \( \Omega_0 \) such that \( B_R(x_0) \subset \Omega_0 \subset \Omega \), let us set

\[ M_{\Omega_0} = \sup_{\Omega_0} u, \quad \text{and} \quad m_{\Omega_0} = \inf_{\Omega_0} u . \]

Let

\[ z_j = (x_0, m_{\Omega_0} + 2jR) , \]
for $j \in \mathbb{N}$. Then
\[ z_j \in U, \]
for
\[ j \leq j_1 = \left\lfloor \frac{u(x_0) - m_{\Omega_1}}{2R} \right\rfloor, \]
where $[s]$ denotes the largest integer less than $s$ for $s > 0$. From (0.3), we have
\[ |U_{R/2}(z_j)| \geq a_* \left( \frac{R}{2} \right)^{n+1}, \]
for $1 \leq j \leq j_1$ and therefore
\[ \int_{B_R(x_0)} (u - m_{\Omega_1}) \, dx \geq \sum_{j=1}^{j_1} |U_{R/2}(z_j)| \geq j_1 a_* \left( \frac{R}{2} \right)^{n+1}. \]

Hence
\[ M_{\Omega_0} = u(x_0) + (M_{\Omega_1} - u(x_0)) \leq 2(j_1 + 1) R + m_{\Omega_1} + (M_{\Omega_0} - u(x_0)) \]
\[ \leq \frac{2^{n+2}}{a_* R^n} \int_{B_R(x_0)} (u - m_{\Omega_1}) \, dx + 2R + m_{\Omega_1} + (M_{\Omega_0} - u(x_0)), \]
that is,
\[ u(x_0) - m_{\Omega_1} \leq \frac{2^{n+2}}{a_* R^n} \int_{B_R(x_0)} (u - m_{\Omega_1}) \, dx + 2R. \] (1.1)

To estimate the integral on the right hand side of (1.1) under the hypotheses that $x_1 \in B_{R^*}(x_0)$ and $B_{2A^*R}(x_1) \subset D_{\beta}$, let $\bar{x}_1 \in \partial B_{2A^*R}(x_1)$ be a point at which
\[ u(\bar{x}_1) - u(x_1) \geq 2A^* R_{\beta}. \]

Let
\[ z_1 = (x_1, u(x_1)). \]
and

$$\tilde{z}_1 = (\tilde{x}_1, u(\tilde{x}_1)).$$

From (0.3) and (0.4), we have

$$|U_{A_{\tilde{x}, \beta} R}(z_1) | \geq \alpha \ast (A_{\tilde{x}, \beta}^R)^n,$$

and

$$|U_{A_{\tilde{x}, \beta} R}(\tilde{z}_1) | \geq \alpha \ast (A_{\tilde{x}, \beta}^R)^n.$$ 

These yield

$$\{ x : x \in B_R(x_0), u(x) \leq u(x_1) + A_{\tilde{x}, \beta} R \} \geq \alpha \ast (A_{\tilde{x}, \beta} R)^n,$$

and

$$\{ x : x \in B_R(x_0), u(x) \geq u(x_1) + A_{\tilde{x}, \beta} R \} \geq \alpha \ast (A_{\tilde{x}, \beta} R)^n.$$

Hence, by Proposition 2, we have

$$(1.2) \quad \int_{B_R(x_0)} (u - m_{\Omega_0}) \, dx \leq \quad \quad \quad \quad \int_{B_R(x_0)} |Du| \, dx + \left( (u(x_1) - m_{\Omega_0}) + A_{\tilde{x}, \beta} R \right) |B_R(x_0)|,$$

with $$C_{\alpha, \beta}$$ being as in (0.8) by setting $$\alpha_1 = \alpha_2 = \frac{\alpha \ast (A_{\tilde{x}, \beta} R)^n}{\omega_x}$$ in (0.9).

Inserting this into (1.1), we obtain (0.5) with the value $$\xi_{u, \alpha}$$ given in (0.7).

1.2. Analogously, we let

$$z_j^+ = (x_0, M_{\Omega_0} - 2Rj),$$

for $$j \in \mathbb{N}$$. Then

$$z_j^+ \in U' = Q \setminus U$$

for

$$j \leq j_1^+ = \left[ \frac{M_{\Omega_0} - u(x_0)}{2R} \right].$$
We obtain from (0.4) that
\[ |U_{2,2}^j(z_1^+) | \geq \alpha_* \left( \frac{R}{2} \right)^{\alpha + 1}, \]
and therefore
\[ \int_{B_R(x_0)} (M_{\Omega_0} - u) \, dx \geq \sum_{j=1}^m |U_{2,2}^j(z_1^+) | \geq j_1^+ \alpha_* \left( \frac{R}{2} \right)^{\alpha + 1}, \]
which yields
\begin{align*}
-m_{\Omega_0} &= u(x_0) + (u(x_0) - m_{\Omega_0}) \\
&\leq -M_{\Omega_0} + 2(j_1^+ + 1) R + (u(x_0) - m_{\Omega_0}) \\
&\leq \frac{2^{\alpha + 2}}{\alpha_* R^{\alpha}} \int_{B_R(x_0)} (M_{\Omega_0} - u) \, dx + 2R - M_{\Omega_0} + (u(x_0) - m_{\Omega_0}).
\end{align*}
That is,
\begin{equation}
(1.4) \quad M_{\Omega_0} - u(x_0) \leq \frac{2^{\alpha + 2}}{\alpha_* R^{\alpha}} \int_{B_R(x_0)} (M_{\Omega_0} - u) \, dx + 2R.
\end{equation}

Under the hypotheses that \( x_1 \in B_{R^*}(x_0) \) and \( B_{2,4^+ R}(x_1) \subset D_\beta \), let \( x_1^+ \in \partial B_{2,4^+ R}(x_1) \) be a point at which
\[ u(x_1) - u(\tilde{x}_1^+) \geq 2A_{R^*,\beta}^+ R\beta, \]
where the point \( x_1 \in B_{R^*}(x_0) \) is chosen as in 1.1. Then setting
\[ \tilde{z}_1^+ = (x_1^+, u(\tilde{x}_1^+)) , \]
we obtain from (0.3) and (0.4) that
\[ |U_{1,2}^j(R)(z_1) | \geq \alpha_* (A_{R^*,\beta}^+ R)^{\alpha + 1} \]
and
\[ |U_{1,2}^j(R)(\tilde{z}_1^+) | \geq \alpha_* (A_{R^*,\beta}^+ R)^{\alpha + 1}, \]
which and Proposition 2 yield
\[
\int_{B_R(x_0)} (M_{\Omega_0} - u) \, dx \leq C_{\alpha_+} R \int_{B_R(x_0)} |Du| \, dx + \\
+ \left( (M_{\Omega_0} - u(x_1)) + A^+_{R, \beta} R \right) |B_R(x_0)|.
\]
This and (1.4) yield (0.6).


2.1. Proof of theorem 2.

Setting \( \varepsilon = \frac{1}{2 \overline{\mathcal{V}}_{\Omega_\delta}^2} \), we obtain from (0.21) that
\[
\int_{\Sigma_\varepsilon} \left| \cos \theta \eta \, d\mathcal{H}^n_{\Omega_\delta} \right| \leq \bar{\gamma} \int_{\Sigma_\varepsilon} |D\eta| \, dx + 2 \bar{\gamma} \overline{\mathcal{K}}_{\Omega_\delta} (n + \delta) \int_{\Sigma_\varepsilon} |\eta| \, dx,
\]
for each \( \delta, 0 < \delta \leq 1 \) and for each \( \eta \in C^1(\overline{\Omega}) \). By this and (0.13), if \( u \) is a solution in (0.1) and (0.2) in the weak sense, we obtain
\[
(2.1) \int_{\Omega} \frac{|Du|}{\sqrt{1 + |Du|^2}} \, D\eta \, dx \leq \bar{\gamma} \int_{\Sigma_\varepsilon} |D\eta| \, dx + \\
+ 2n \bar{\gamma} \overline{\mathcal{K}}_{\Omega_\delta} \int_{\Sigma_\varepsilon} |\eta| \, dx - n \int_{\Omega} H\eta \, dx,
\]
for each \( \eta \in C^1(\Omega) \). Taking \( \eta = u - m_{\Omega_\delta} \geq 0 \), we obtain from
\[
\frac{|Du|}{\sqrt{1 + |Du|^2}} = \sqrt{1 + |Du|^2} - \frac{1}{\sqrt{1 + |Du|^2}} > |Du| - 1
\]
and (2.1) that
\[
(2.2) \int_{\Omega \setminus \Sigma_{\varepsilon}} |Du| \, dx + (1 - \bar{\gamma}) \int_{\Sigma_{\varepsilon}} |Du| \, dx < \\
< |\Omega| + 2n \bar{\gamma} \overline{\mathcal{K}}_{\Omega_\delta} \int_{\Sigma_{\varepsilon}} (u - m_{\Omega_\delta}) \, dx - n \int_{\Omega} H(u - m_{\Omega_\delta}) \, dx.
\]
Taking $\eta = M_{\Omega_0} - u$ in (2.1) instead, we obtain
\begin{equation}
(2.3) \quad \int_{\omega \Sigma_{\Omega_0}} |Du| \, dx + (1 - \gamma) \int_{\Sigma_{\Omega_0}} |Du| \, dx <
\end{equation}
\begin{equation*}
< |\Omega| + 2 n \gamma \chi_{\Sigma_{\Omega}} \int_{\Sigma_{\Omega_0}} (M_{\Omega_0} - u) \, dx - n \int_{\Omega} H(M_{\Omega_0} - u) \, dx.
\end{equation*}

In case (0.22) holds, we obtain from (2.2) and (2.3) that
\begin{equation}
(2.4) \quad \int_{\omega \Sigma_{\Omega_0}} |Du| \, dx < |\Omega|.
\end{equation}
and
\begin{equation}
(2.5) \quad \int_{\Omega} |Du| \, dx < \frac{1}{1 - \gamma} |\Omega|.
\end{equation}

Inserting (2.5) into (0.5) and (0.6), we obtain respectively (0.23) and (0.24). By using (2.4) instead of (2.5) and replacing $R$ by $R/2$ in (0.5) and (0.6), we obtain (0.25), (0.26), (0.27) and (0.28).

2.2. Proof of corollary 1.

Suppose that $u$ is a solution to (0.1) and (0.2) in the weak sense in $B_R(x_0)$ and that (0.22*) holds. We insert the inequality (0.30) into (0.13) and take $\eta = u - m_R$ and $\eta = M_R - u$ in (0.12) to obtain, respectively
\begin{equation}
(2.6) \quad \int_{B_R(x_0)} |Du| \, dx \leq |B_R(x_0)| +
\end{equation}
\begin{equation*}
+ n \omega_a \gamma R a^{-1} (M_R - m_R) - n \int_{B_R(x_0)} H(u - m_R) \, dx,
\end{equation*}
and
\begin{equation}
(2.7) \quad \int_{B_R(x_0)} |Du| \, dx \leq |B_R(x_0)| +
\end{equation}
\begin{equation*}
+ n \omega_a \gamma R a^{-1} (M_R - m_R) - n \int_{B_R(x_0)} H(M_R - u) \, dx.
\end{equation*}
Since $M_R - m_R = (M_R - u(x_0)) + (u(x_0) - m_R)$, we have either

$$\int_{B_R(x_0)} H(u - m_R) \, dx \geq \frac{1}{2} \left( \inf_{x \in B_R(x_0)} H(M_R - m_R) B_R(x_0) \right),$$

or

$$\int_{B_R(x_0)} H(M_R - u) \, dx \geq \frac{1}{2} \left( \inf_{x \in B_R(x_0)} H(M_R - m_R) B_R(x_0) \right).$$

Inserting these and (0.22*) into (2.6) and (2.7) yields estimates of $\int_{B_R(x_0)} |Du| \, dx$, which we subsequently insert into either (0.5) or (0.6) to establish Corollary 1.

2.3. Proof of corollary 2.

Suppose that $u$ is a solution to (0.1) and (0.2) in the weak sense in $B_R(x_0)$ and that (0.34) holds. We obtain from (2.6), (0.12) and (0.34)

$$\int_{B_R(x_0)} |Du| \, dx \leq \frac{r(M_R - m_R)}{\xi_{\alpha, \alpha} C_{\alpha, \beta}},$$

Inserting this into (0.5) and (0.6), we obtain respectively

(2.8) \hspace{1cm} u(x_0) - m_R \leq r(M_R - m_R) + \xi_{\alpha, \alpha} \omega_u(u(x_1) - m_R) + \xi u_{2, R}

and

(2.9) \hspace{1cm} M_R - u(x_0) \leq r(M_R - m_R) + \xi_{\alpha, \alpha} \omega_u(M_R - u(x_1)) + \xi u_{2, R},

where $u_{2, R}$ is given in (0.33). If $u(x_0) - m_R \leq M_R - u(x_0)$, we have $M_R - m_R \leq 2(u(x_0) - m_R)$, which and (2.8) yield (0.35). If $M_R - u(x_0) \leq u(x_0) - m_R$, then we have $M_R - m_R \leq 2(M_R - u(x_0))$, which and (2.9) yield (0.36).
3. Proof of Harnack’s inequality for solutions to the mean curvature equation and proof of a boundary Harnack’s inequality.

Choose \( \eta_{\lambda} = \eta_{\lambda}(q) \in C^1(B_R(x_0)) \), \( q = \text{dist}(x_0, x_1) \), with

\[
\begin{align*}
0 \leq \eta_{\lambda} &\leq 1, \\
\eta_{\lambda} &\equiv 1 \quad \text{in } B_{\lambda R}(x_0), \\
\eta_{\lambda} &\equiv 0 \quad \text{on } \partial B_R(x_0),
\end{align*}
\]

for some \( \lambda, \, 0 < \lambda < 1 \), such that

\[
\frac{1}{(1 - \lambda) R} \leq |D\eta_{\lambda}(q)| \leq \frac{1 + \delta_0}{(1 - \lambda) R},
\]

for some \( \delta_0, \, 0 < \delta_0 < 1 \) and for \( \lambda R \leq q \leq R \). Thus

\[
1 - \frac{q}{R} \leq \eta_{\lambda}(q) \leq 1 - \frac{(1 + \delta_0) q}{R},
\]

for \( \lambda \leq q \leq R \).

Suppose that \( u \in C^2(\Omega) \) is a solution to (0.1) in \( \Omega \). Taking \( \eta = \eta_{\lambda}(u - m_R) \) and \( \eta = \eta_{\lambda}(M_R - u) \) in (0.12), we obtain

\[
\int_{B_R(x_0)} \frac{|Du|}{\sqrt{1 + |Du|^2}} \cdot D\eta \, dx + n \int_{B_R(x_0)} H \eta \, dx = 0,
\]

which yields

\[
\int_{B_{\lambda R}(x_0)} \frac{|Du|^2}{\sqrt{1 + |Du|^2}} \, dx \leq
\]

\[
\leq \int_{B_R(x_0) \setminus B_{\lambda R}(x_0)} |D\eta_{\lambda}| (u - m_R) \, dx - n \int_{B_R(x_0)} H \eta_{\lambda}(u - m_R) \, dx =
\]

\[
= \int_{B_R(x_0) \setminus B_{\lambda R}(x_0)} |D\eta_{\lambda}| (u - m_R) \, dx - n \int_{B_R(x_0) \setminus B_{\lambda R}(x_0)} H \eta_{\lambda}(u - m_R) \, dx -
\]

\[
- n \int_{B_{\lambda R}(x_0)} H(u - m_R) \, dx,
\]
and

\begin{align*}
\int_{B_R(x_0)} \frac{|Du|^2}{1 + |Du|^2} \, dx &\leq \int_{B_R(x_0) \setminus B_\frac{1}{2}R(x_0)} |D\eta_\lambda|(M_R - u) \, dx - n \int_{B_R(x_0) \setminus B_\frac{1}{2}R(x_0)} H\eta_\lambda(M_R - u) \, dx = \\
&\leq \int_{B_R(x_0) \setminus B_\frac{1}{2}R(x_0)} |D\eta_\lambda|(M_R - u) \, dx - n \int_{B_R(x_0) \setminus B_\frac{1}{2}R(x_0)} H\eta_\lambda(M_R - u) \, dx - n \int_{R_\frac{1}{2}R(x_0)} H(M_R - u) \, dx.
\end{align*}

By (3.1) and (3.2), we have

\begin{align*}
\int_{B_R(x_0) \setminus B_\frac{1}{2}R(x_0)} |D\eta_\lambda|(M_R - m_R) \, dx &\leq w_\lambda(M_R - m_R) \frac{1 + \delta_0}{(1 - \lambda) R} \int_{\frac{1}{2}R} q^{n-1} \, dq = \\
&= \left( \frac{1 + \delta_0}{1 - \lambda} \right) (1 - \lambda^n)(M_R - m_R) w_\lambda R^{n-1}.
\end{align*}

Since \( \delta_0 \) can be arbitrarily small, we have

\begin{equation}
\int_{B_R(x_0) \setminus B_\frac{1}{2}R(x_0)} |D\eta_\lambda|(M_R - m_R) \, dx \leq \left( \frac{1 - \lambda^n}{1 - \lambda} \right)(M_R - m_R) w_\lambda R^{n-1}.
\end{equation}

3.1. Proof of theorem 3.

We also have

\begin{align*}
n \int_{B_R(x_0) \setminus B_\frac{1}{2}R(x_0)} H\eta_\lambda(M_R - m_R) \, dx &\geq \\
\geq n^2 \left( \inf_{x \in B_R(x_0)} H \right) w_\lambda(M_R - m_R) \left[ \int_{\frac{1}{2}R} \left( 1 - \frac{1 + \delta_0}{R} q \right) q^{n-1} \, dq \right] \\
\geq n^2 \left( \inf_{x \in B_R(x_0)} H \right) (M_R - m_R) \left( \frac{1 - \lambda^n}{n} - \frac{1 - \lambda^{n+1}}{n + 1} \right) w_\lambda R^n.
\end{align*}
Since $\delta_0$ can be arbitrarily small, we obtain
\[
(3.7) \quad n \int_{B_R(x_0) \setminus B_M(x_0)} H \eta(J(M_R - m_R)) \, dx \geq \n^2 \left( \inf_{x \in B_R(x_0)} H \right) (M_R - m_R) \left( \frac{1 - \lambda^n}{n} - \frac{1 - \lambda^{n+1}}{n+1} \right) \omega_n R^{n-1}.
\]
Moreover, we have
\[
(3.8) \quad n \int_{B_R(x_0)} H(M_R - m_R) \, dx \geq n \left( \inf_{x \in B_R(x_0)} H \right) (M_R - m_R) \lambda^n \omega_n R^n.
\]
From (3.6), (3.7), (3.8) and (0.39), we obtain
\[
\int_{B_R(x_0) \setminus B_M(x_0)} |D \eta(J(M_R - m_R)) \, dx -
-n \int_{B_R(x_0) \setminus B_M(x_0)} H \eta(J(M_R - m_R)) \, dx - n \int_{B_M(x_0)} H(M_R - m_R) \, dx \leq
\leq (M_R - m_R) \omega_n R^{n-1},
\]
\[
\left( \frac{1 - \lambda^n}{1 - \lambda} \right) - n^2 \left( \inf_{x \in B_R(x_0)} H \right) R \left( \frac{1 - \lambda^n}{n} - \frac{1 - \lambda^{n+1}}{n+1} + \lambda^n \right) =
\leq (M_R - m_R) \omega_n R^{n-1},
\]
\[
\left( \frac{1 - \lambda^n}{1 - \lambda} \right) \left[ 1 - n \left( \inf_{x \in B_R(x_0)} H \right) R \left( 1 - \lambda \right) \left( 1 - \frac{n}{n+1} \frac{1 - \lambda^{n+1}}{1 - \lambda^n} + \frac{\lambda^n}{1 - \lambda^n} \right) \right] =
\leq (M_R - m_R) \omega_n R^{n-1} C_\lambda.
\]
\[
\left[ 1 - \left( \inf_{x \in B_R(x_0)} H \right) R \left( \frac{n}{n+1} \right) (1 - \lambda) \left( 1 - \frac{n\lambda^n}{C_\lambda} \right) - \left( \inf_{x \in B_R(x_0)} H \right) R \left( \frac{n\lambda^n}{C_\lambda} \right) \right] =
\leq (M_R - m_R) \omega_n R^{n-1} C_\lambda.
\]
\[
\left[ 1 - \left( \inf_{x \in B_R(x_0)} H \right) R \left( \frac{n}{n+1} \right) (1 - \lambda) \left( 1 - \frac{\lambda^n}{C_\lambda} \right) - n \left( \inf_{x \in B_R(x_0)} H \right) R \frac{\lambda^{n+1}}{C_\lambda} \right].
\]
Since there holds either $M_R - m_R \leq 2(u(x_1) - m_R)$ or $M_R - m_R \leq 2(M_R - u(x_1))$, we have either

$$\int_{B_R(x_0) \setminus B_{\lambda R}(x_0)} |D\eta| (u - m_R) \, dx - n \int_{B_R(x_0)} H\eta (u - m_R) \, dx \leq 2(u(x_1) - m_R) C_* C^{\#} \omega_n R^{n-1},$$

or

$$\int_{B_R(x_0) \setminus B_{\lambda R}(x_0)} |D\eta| (M_R - u) \, dx - n \int_{B_R(x_0)} H\eta (M_R - u) \, dx \leq 2(M_R - u(x_1)) C_* C^{\#} \omega_n R^{n-1}.$$

where $C_1$ and $C_1^\#$ are given respectively in (0.39) and (0.40). Inserting these into (3.4) or (3.5), and subsequently insert what results in into (0.5) or (0.6), we obtain (0.37) and (0.38).

Since there also holds either $M_R - m_R \leq 2(u(x_0) - m_R)$ or $M_R - m_R \leq 2(M_R - u(x_0))$, we have, either

$$\int_{B_R(x_0) \setminus B_{\lambda R}(x_0)} |D\eta| (u - m_R) \, dx - n \int_{B_R(x_0)} H\eta (u - m_R) \, dx \leq 2(u(x_0) - m_R) C_* C^{\#} \omega_n R^{n-1},$$

or

$$\int_{B_R(x_0) \setminus B_{\lambda R}(x_0)} |D\eta| (M_R - u) \, dx - n \int_{B_R(x_0)} H\eta (M_R - u) \, dx \leq 2(M_R - u(x_0)) C_* C^{\#} \omega_n R^{n-1}.$$

Inserting these into (3.4) or (3.5), and subsequently insert what results in into (0.5) or (0.6), we obtain (0.42) and (0.43) under the hypothesis of (0.41).

Since there holds either \(M_R - m_R \leq 2(u(x_1) - m_R)\) or \(M_R - m_R \leq 2(M_R - u(x_1))\), we obtain from (3.6) that either

\[
\int_{R_g(x_0) \setminus R_{\delta g}(x_0)} |D\eta_1| (u(x_0) - m_R) \, dx \leq 2 \left( \frac{1 - \lambda^2}{1 - \lambda} \right) (u(x_1) - m_R) \omega_g R_n^{n-1} = 2C_1 (u(x_1) - m_R) \omega_g R_n^{n-1},
\]

or

\[
\int_{R_g(x_0) \setminus R_{\delta g}(x_0)} |D\eta_1| (M_R - u(x_0)) \, dx \leq 2 \left( \frac{1 - \lambda^2}{1 - \lambda} \right) (u(x_1) - m_R) \omega_g R_n^{n-1} = 2C_1 (M_R - u(x_1)) \omega_g R_n^{n-1},
\]

Inserting these into (3.4) or (3.5), and subsequently insert what results in into (0.5) or (0.6), we obtain (0.48) and (0.49).

Since there also holds either \(M_R - m_R \leq 2(u(x_0) - m_R)\) or \(M_R - m_R \leq 2(M_R - u(x_0))\), we have, either

\[
\int_{R_g(x_0) \setminus R_{\delta g}(x_0)} |D\eta_1| (u(x_0) - m_R) \, dx \leq 2 \left( \frac{1 - \lambda^2}{1 - \lambda} \right) (m_R - u(x_0)) \omega_g R_n^{n-1} = 2C_1 (u(x_0) - m_R) \omega_g R_n^{n-1},
\]

or

\[
\int_{R_g(x_0) \setminus R_{\delta g}(x_0)} |D\eta_1| (M_R - u(x_0)) \, dx \leq 2 \left( \frac{1 - \lambda^2}{1 - \lambda} \right) (M_R - u(x_0)) \omega_g R_n^{n-1} = 2C_1 (M_R - u(x_1)) \omega_g R_n^{n-1}.
\]

Inserting these into (3.4) or (3.5), and subsequently insert what results in into (0.5) or (0.6), we obtain (0.51) and (0.52) under the hypothesis of (0.50).

The equation (0.1) is the Euler equation of the functional
\[ \mathcal{F}_\beta(v) = \int_\Omega \sqrt{1 + |Dv|^2} dx + n \int_0^v H(x, t) \, dt \, dx . \]

And corresponding to the Dirichlet problem with boundary data \( \psi \) and the capillarity problem with boundary contact angle \( \theta \) are the problems of minimizing the respective functionals
\[ \mathcal{F}_\beta(v) + \int_{\partial \Omega} |v - \psi| \, d\mathcal{H}_{n-1} \]
and
\[ \mathcal{F}_\beta(v) + \int_{\partial \Omega} (\cos \theta) \, v \, d\mathcal{H}_{n-1} \]
among all \( v \in BV(\Omega) \), where \( \mathcal{H}_k \) is the \( k \)-dimensional Hausdorff measure.

Alternatively, we consider the problem of minimizing the functional
\[ \mathcal{F}(v) = \int_\Omega \sqrt{1 + |Du|^2} dx + \int_\Omega H(x, t) \, dx \, dt + \int_{\partial \Omega} \kappa(x, v) \, d\mathcal{H}_{n-1} , \]
with
\[ \kappa(x, v) = \int_0^v \gamma(x, t) \, dt . \]

For the capillarity problem, we have
\[ \gamma(x, t) = \cos \theta \]
and
\[ \kappa(x, t) = \int_0^v \cos \theta \, dt . \]

For the Dirichlet problem, we have
\[ \gamma(x, t) = 1 - 2\phi(x, t) \]
and

\[ \kappa(x, u) = |u - f(x)| - |f(x)|. \]

Here and throughout this section, \( \phi_V \) is the characteristic function of the subgraph \( V \) of \( v \):

\[ \phi_V(x, t) = \begin{cases} 1, & \text{if } t < v(x), \\ 0, & \text{if } t \geq v(x). \end{cases} \]

M. Miranda [22] introduced the notion of \textit{generalized solutions} for the minimal surface equation and used it successively both in the Dirichlet problem in infinite domains [22] and in the problem of removable singularities [23]. E. Giusti in [11] and [12] used the same notion of generalized solutions respectively in the problem of maximal domains for the mean curvature equation and boundary value problems for the mean curvature equation.

The idea of \textit{generalized solutions} originates from the observation that a function \( u : \Omega \rightarrow \mathbb{R} \) is a solution of (0.1) in \( \Omega \) if and only if its subgraph

\[ U = \{ (x, t) \in \Omega \times \mathbb{R} : t < u(x) \} \]

minimizes the functional

\[ F_s(U) = \int_{\Omega \times \mathbb{R}} |D\phi_V| + n \int_{\Omega \times \mathbb{R}} H\phi_V \, dx \, dt \]

locally in \( \Omega \times \mathbb{R} \), in the sense that for every set \( V \) coinciding with \( U \) outside some compact set \( K \subset \Omega \times \mathbb{R} \), we have

\[ \int_k |D\phi_V| + n \int_k H\phi_V \, dx \, dt \leq \int_k |D\phi_V| + n \int_k H\phi_V \, dx \, dt. \]

Moreover, a function \( u \in BV(\Omega) \) minimizes \( \mathcal{F} \) in \( \Omega \) if and only if its subgraph minimizes the functional

\[ F(U) = \int_{\Omega \times \mathbb{R}} |D\phi_V| + n \int_{\Omega \times \mathbb{R}} H\phi_V \, dx \, dt + \int_{\partial \Omega \times \mathbb{R}} \gamma \phi_V \, d\mathcal{H}_n. \]

Minimization is here to be understood in the following sense: for \( T > 0 \), set

\[ Q_T = \Omega \times [-T, T], \quad \partial Q_T = \partial \Omega \times [-T, T], \]
and for $U \subset Q$,
\[
F_T(U) = \int_{Q_T} |D\phi_U| + n \int_{Q_T} H\phi_U \, dx \, dt - \int_{\partial Q_T} \gamma\phi_U \, d\mathcal{H}^n.
\]
We say that $U$ minimizes $F_T$ in $Q_T$ if
\[
F_T(U) \leq F_T(S)
\]
for every Caccioppoli set $S \subset Q_T$. We say that $U$ minimizes $F$ in $\Omega \times \mathbb{R}$ if $U$ minimizes $F_T$ in $Q_T$ for every $T > 0$.

**Definition (Miranda[22]).**

1. A function $u : \Omega \rightarrow [-\infty, \infty]$ is a generalized solution of the equation (0.1) in $\Omega$ if its subgraph $U$ minimizes the functional $F_*$ locally in $\Omega \times \mathbb{R}$.
2. A function $u : \Omega \rightarrow [-\infty, \infty]$ is a generalized solution for the function $F$ if its subgraph $U$ minimizes $F$ in $Q$.

We note that a generalized solution can take the values $\pm \infty$ on a set of positive $n$-dimensional Hausdorff measure. However, it follows from Miranda [21] that if a generalized solution $u(x)$ can be modified on a set of zero $n$-dimensional Hausdorff measure to be locally bounded, then $u(x)$ is a classical solution of (0.1) in $\Omega$.

Proposition 5 below is derived in the proof of Theorem 1.1 of Giusti [12]. The special case in Proposition 6 below where $\mu = 1$ and $\partial \Omega \in C^2$ are shown by the proof of Theorem 3.2 of [12]. Proposition 6 is fully established in Lemma 7.6 of Finn [3].

It is easy to see that Proposition 3 and Proposition 4 follow immediately from Proposition 5 and Proposition 6, together with a subsequent consideration of $U'$ instead of $U$.

**Proposition 5.** Let $U$ minimize $F_*$ locally in $Q = \Omega \times \mathbb{R}$. If $z_0 = (x_0, t_0)$ is a point in $Q$ and if for all $r > 0$ we have
\[
|U_r(z_0)| > 0,
\]
then, there exist positive constants $C_0$ and $R_0$, depending only on $n$ and $\inf \mathcal{H}^n$ such that
\[
|U_r(z_0)| \geq C_0 r^{n+1},
\]
(A.1)
for every } \ r \leq \min (R_0, \ \text{dist}(z_0, \ \partial Q)), \ \text{where we set}
\[ U_r(z_0) = U \cap C_r(z_0), \]
with
\[ C_r(z_0) = \{ z = (x, t) : |x - x_0| < r, \ |t - t_0| < r \}. \]
In particular, we can take
\[ C_0 = \frac{1}{4(n + 1) k_{(n+1)}}, \]
and
\[ R_0 = \begin{cases} \left( \frac{1}{2nk_{(n)} \omega_n | \inf_{Q} H(x, t) |} \right)^{1/n} & \text{if } \inf_{Q} H(x, t) < 0, \\ \infty & \text{if } \inf_{Q} H(x, t) \geq 0, \end{cases} \]
where we denote } k_{(n)} \text{ the isoperimetric constant in } \mathbb{R}^m, \ m \geq 1.

**Proposition 6.** Suppose that there exist constants } Q_0 > 0 \text{ and } \bar{\gamma}, \ \ 0 < \bar{\gamma} < 1, \ \text{such that}
\[ \gamma(x, t) \geq -\bar{\gamma}, \ \text{for all } x \in \partial \Omega \text{ and } t > \theta_0. \]
Suppose further that for some constant } \mu, \ \text{with } \mu \bar{\gamma} < 1 \text{ and } C_\Omega \ \text{depending only on } \Omega, \ \text{an inequality}
\[ \left( \int_{\partial \Omega} |v| dx \right) \leq \mu \left( \int_{\partial \Omega} |Dv| dx + C_\Omega \int_{\partial \Omega} |v| dx \right), \]
holds for all } v \in \text{BV}(\Omega). \ \text{Let } U \text{ minimizes } F \text{ in } Q = \Omega \times R, \ \text{and let } z_0 = (x_0, t_0), \ t_0 > \theta_0 + 1, \ \text{be a point of } \partial Q \ \text{such that for every positive } r
\[ |U_r| > 0, \]
where } U_r \text{ is defined as in Proposition 1. Then there exist constants } R_1 > 0 \text{ and } C_1 > 0 \ \text{determined completely by } n, \ \inf_{Q} H(x, t), \ \bar{\gamma}, \ \mu \ \text{and } C_\Omega \ \text{such that}
\[ |U_r| \geq C_1 r^{n+1}, \ \text{for every } r \leq R_1. \]
In particular, we can take

$$C_1 = \frac{1 - \gamma}{16(n + 1) k(n+1)}.$$  

and if $\inf_{\mathcal{Q}} H(x, t) \geq 0$,

$$R_1 = \left( \frac{C^2}{C_k k(n+1)(2 \omega_n)\gamma^{\frac{1}{n+1}}} \right),$$  

where we set

$$C^* = \min\left( \frac{1}{2}, \frac{1 - (2\mu - 1) \gamma}{3 \gamma + 1} \right);$$

if $\inf_{\mathcal{Q}} H(x, t) < 0$, we firstly take $\bar{R}_1$ so small that

$$\bar{R}_1 \leq \left( \frac{1 - (2\mu - 1) \gamma}{2(\mu + 1) \nu k(n) \omega_n \inf_{\mathcal{Q}} |H|} \right)^{1/n},$$

and then take

$$R_1 = \min\left( \frac{C^{**}}{C_k k(n+1)(2 \omega_n)\gamma^{\frac{1}{n+1}}}, \bar{R}_1 \right),$$  

where we set

$$C^{**} = \min\left( \frac{1}{2}, \frac{1 - (2\mu - 1) \gamma - (\mu + 1) \nu k(n) \inf_{\mathcal{Q}} |H| \omega_n (\bar{R}_1)^{\gamma}}{3 \gamma + 1} \right).$$

We notice that Lemma 1.1 in Giusti [11] established (A.4) for $\mu = 1$ in the special case that $\partial \Omega \in C^2$ and we have formulated this result as Lemma 1 in 0.2 of our present work.

An inequality of the form (A.4) appears first in Emmer [2], with

$$\mu = \sqrt{1 + L^2}$$

for any Lipschitz domain with Lipschitz constant $L$. (See also [19, page 203]). On pages 141-143 of Finn [3], this result is extended to include domains in which one or more corners with inward opening angle appear. As pointed out on page 197 of [3], this extended result permits inward cusps and even boundary segments that may physically coincide but are
adjacent to different parts of $\Omega$. However, it is pointed out on page 143 of [3] that an outward cusp or a vertex of an outward corner is not permitted.

We end this section with a sketch of the reasoning in [3] and [12] which leads to Propositions 5 and 6, mainly for the purpose of unifying the notation designations in [3] and [12]. Indeed, from comparing the values of the functionals $F_*$ and $F$ taken by $U$ with those taken by $U \setminus C_r$, we obtain

$$
\int_{C_r} |D\phi_U| + n \int_{C_r} H\phi_U \, dx \, dt \leq \int_{\partial C_r} \phi_U \, d\mathcal{H}^n, \quad \text{if } r < \text{dist}(z_0, \partial \Omega),
$$

and

$$
\int_{Q \cap C_r} |D\phi_U| + n \int_{Q \cap C_r} H\phi_U \, dx \, dt + \int_{\partial Q \cap C_r} \gamma \phi_U \, d\mathcal{H}^n \leq \int_{\partial C_r} \phi_U \, d\mathcal{H}^n, \quad \text{if } r \geq \text{dist}(z_0, \partial \Omega).
$$

Since

$$
\int_{Q} |D\phi_U| = \int_{C_r} |D\phi_U| + \int_{\partial C_r} \phi_U \, d\mathcal{H}^n
$$

for almost all $r$, the previous two inequalities lead respectively to

(A.9) \quad \int_{Q} |D\phi_U| + n \int_{Q} H\phi_U \, dx \, dt \leq 2 \int_{\partial C_r} \phi_U \, d\mathcal{H}^n, \quad \text{if } r < \text{dist}(z_0, \partial \Omega),

and

(A.10) \quad \int_{Q} |D\phi_U| + n \int_{Q} H\phi_U \, dx \, dt + \int_{\partial Q} \gamma \phi_U \, d\mathcal{H}^n \leq 2 \int_{\partial C_r} \phi_U \, d\mathcal{H}^n, \quad \text{if } r \geq \text{dist}(z_0, \partial \Omega).

Setting

$$
H_0^-(x) = \min \left( \inf_t H(x, t), 0 \right),
$$
the curvature term can be estimated as follows:

\[(A.11) \quad \int_{Q} H \phi \, dx \, dt \geq \int_{Q} H_{0}^{-} \phi \, dx \, dt \]
\[
\geq -\|H_{0}^{-}\|_{\infty,B_{r}(z_{0})} \int_{t_{0} - r}^{t_{0} + r} \left| C_{r} \right|^{1 - \frac{2}{n}} \, dt, \quad \text{by Hölder's inequality}
\]
\[
\geq -k(n)\|H_{0}^{-}\|_{\infty,B_{r}(z_{0})} \int_{t_{0} - r}^{t_{0} + r} \left( \int \left| D\phi_{C_{r}} \right| \right) \, dt, \quad \text{by the isoperimetric inequality}
\]
\[
\geq -k(n)\|H_{0}^{-}\|_{\infty,B_{r}(z_{0})} \int \left| D\phi_{r} \right|.
\]

Inserting this into (A.9), we obtain, if \(r < \text{dist}(z_{0}, \partial Q)\),

\[(A.12) \quad (1 - nk(n))\|H_{0}^{-}\|_{\infty,B_{r}(z_{0})} \int \left| D\phi_{r} \right| \leq 2 \int_{\partial C_{r}} \phi \, d\mathcal{H}^{n},
\]

which yields

\[
\frac{d}{dr} |U_{r}| = \int_{\partial C_{r}} \phi \, d\mathcal{H}^{n}
\]
\[
\geq \frac{1}{2} (1 - nk(n))\|H_{0}^{-}\|_{\infty,B_{r}(z_{0})} \int \left| D\phi_{r} \right|
\]
\[
\geq \frac{1 - nk(n)}{2k(n+1)} \|H_{0}^{-}\|_{\infty,B_{r}(z_{0})} \left( \frac{U_{r}}{r^{n+1}} \right), \quad \text{again by the isoperimetric inequality},
\]
\[
\geq \frac{1}{4k(n+1)} \left| U_{r} \right|^{\frac{n}{n+1}},
\]

if we choose \(r\) so small that \(\|H_{0}^{-}\|_{\infty,B_{r}(z_{0})} \leq \frac{1}{2nk(n)}\).

This leads to the estimate (A.1) with \(C_{0}\) taken as in (A.2), whenever \(r < \text{dist}(z_{0}, \partial Q)\), with \(R_{0}\) given in (A.3).

In case \(r \geq \text{dist}(z_{0}, \partial Q)\), we have to handle the third term on the right
hand side of (A.10). By (A.4) and the isoperimetric inequality,
\[
\int_{\Omega} \phi_{U_r} d\mathcal{H}_n \leq \mu \int \frac{|D\phi_{U_r}|}{q} + C_\Omega |U_r| \left\{ \frac{1}{1} \right\} \int |D\phi_{U_r}|.
\]

Since we have
\[
\int |D\phi_{U_r}| d\mathcal{H}_n = \int |D\phi_{U_r}| + \int \phi_{U_r} d\mathcal{H}_n,
\]
the last inequality leads to
\[
(A.13) \quad \int_{\Omega} \phi_{U_r} d\mathcal{H}_n \leq \frac{\mu + C_\Omega k_{(n+1)}}{1 - C_\Omega k_{(n+1)}} \left( \frac{1}{|C_r|^{\frac{1}{n+1}}} \right) \int |D\phi_{U_r}| \leq \frac{\mu + C_\Omega k_{(n+1)}}{1 - C_\Omega k_{(n+1)}} \left( \frac{1}{|C_r|^{\frac{1}{n+1}}} \right) \int |D\phi_{U_r}|,
\]
if \( r \) is so small that
\[
(A.14) \quad C_\Omega k_{(n+1)} |C_r|^{\frac{1}{n+1}} \leq \frac{1}{2}.
\]

This and the last identity yield
\[
(A.15) \quad \int |D\phi_{U_r}| d\mathcal{H}_n \leq \frac{\mu + 1}{1 - C_\Omega k_{(n+1)}} \left( \frac{1}{|C_r|^{\frac{1}{n+1}}} \right) \int |D\phi_{U_r}|.
\]

From (A.11), (A.13) and (A.15), we obtain
\[
(A.16) \quad n \int_{\Omega} H\phi_{U_r} dx \, dt + \int_{\partial \Omega} \gamma \phi_{U_r} d\mathcal{H}_n \geq$
\[
\geq - \left\{ \frac{\mu + C_\Omega k_{(n+1)}}{1 - C_\Omega k_{(n+1)}} |C_r|^{\frac{1}{n+1}} + \frac{(\mu + 1) nk_{(n+1)}}{1 - C_\Omega k_{(n+1)}} |C_r|^{\frac{1}{n+1}} \right\} \int |D\phi_{U_r}|.
\]
Choosing \( r \) so small that (A.7) is satisfied if \( \inf_{\Omega} H \geq 0 \) and (A.8) is satisfied if \( \inf_{\Omega} H < 0 \), we know that (A.14) is satisfied and the right hand side
of (A.16) is bounded below by \( \left( -\frac{(1+\varphi)}{2} \right) \int |D\phi_{U_r}| \). Inserting this into (A.10), we obtain

(A.17) \[
\int_{\mathcal{N}_r} \phi_{U_r} d\mathcal{H}^n \geq \frac{1 - \varphi}{4} \int |D\phi_{U_r}|.
\]

From this, (A.14), (A.15) and the isoperimetric inequality, we obtain

\[
\frac{d}{dr} |U_r| = \int_{\mathcal{N}_r} \phi_{U_r} d\mathcal{H}^n \geq \frac{1 - \varphi}{16} \int |D\phi_{U_r}| \geq \frac{1 - \varphi}{16} \frac{1}{k(n+1)} |U_r|^{\frac{n}{n+1}}.
\]

This leads to the estimate (A.5) with \( C_1 \) taken as in (A.6) and completes the proof of Proposition 6.

REFERENCES


Manoscritto pervenuto in redazione il 25 novembre 2002.