A Note on Abelian Varieties Embedded in Quadrics.

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ABSTRACT - We show that if $A$ is a $d$-dimensional abelian variety in a smooth quadric of dimension $2d$ then $d = 1$ and $A$ is an elliptic curve of bidegree $(2, 2)$ on a quadric. This extends a result of Van de Ven which says that $A$ only can be embedded in $P^{2d}$ when $d = 1$ or 2.

1. Introduction.

Let $A$ be a $d$-dimensional abelian variety embedded in $P^N$. It is well known that $2d \leq N$. Moreover, in [8] Van de Ven proved that the equality holds only when $d = 1$ or 2.

It is a natural question to study the possibilities for $d$ when the abelian variety $A$ is embedded in any other smooth $2d$-dimensional variety $V$. In particular, here we study the embedding in smooth quadrics. We obtain the following result:

THEOREM 1.1. If $A$ is a $d$-dimensional abelian variety in a smooth quadric of dimension $2d$ then $d = 1$ and $A$ is an elliptic curve of bidegree $(2, 2)$ on a quadric.

We will use similar methods to Van de Ven's proof. The calculation of the self intersection of $A$ in the quadric and the Riemann-Roch theorem for abelian varieties allow only the cases $d = 1, 2, 3$.

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The case $d = 1$ is the classical elliptic curve of type $(2, 2)$ contained in the smooth quadric of $\mathbb{P}^3$.

When $d = 2$, $A$ is an abelian surface in $\mathbb{P}^5$. We see that is the projection of an abelian surface $A \subset \mathbb{P}^6$ given by a $(1, 7)$ polarization. By a result [7] due to R. Lazarsfeld, this is projectively normal and it is not contained in quadrics. Therefore, $A$ is not contained in quadrics either.

Finally, two results [5], [6] of J. N. Iyer allow us to discard the case $d = 3$.

2. Proof of the Theorem.

Let $j : A \hookrightarrow Q$ be an embedding of a $d$-dimensional abelian variety into a $2d$-dimensional smooth quadric, with $d > 1$. The Chow ring of the smooth quadric in codimension $d$ is generated by cocycles $a$ and $b$ with the relations $a^2 = b^2 = 1$, $\alpha \beta = 0$. Thus, $A$ will be equivalent to $a^2 + b^2$. (1)

On the other hand, by the self-intersection formula ([4], pag 431) we have $A.A = j^* (\text{NA}, Q)$.

To obtain $c_2(N_A, Q)$, let us consider the normal bundle sequence:

$$0 \rightarrow T_A \rightarrow j^* T_Q \rightarrow N_{A, Q} \rightarrow 0$$

Since the tangent bundle of an abelian variety is trivial, we see that $c(N_{A, Q}) = j^*(c(T_Q))$. We compute the class of the tangent bundle of a quadric in the following lemma:

**LEMMA 2.1.** Let $i : Q \hookrightarrow \mathbb{P}^{n+1}$ be an $n$-dimensional smooth quadric in $\mathbb{P}^{n+1}$. Then

$$c(T_Q) = (1 + H)^{n+2}(1 + 2 \bar{H})^{-1}$$

where $\bar{H} = i^* H$ and $H$ is a hyperplane in $\mathbb{P}^{n+1}$.

**PROOF.** We have an exact sequence:

$$0 \rightarrow T_Q \rightarrow i^* T_{\mathbb{P}^{n+1}} \rightarrow N_{Q, \mathbb{P}^{n+1}} \rightarrow 0$$

Since $Q$ is a hypersurface $N_{Q, \mathbb{P}^{n+1}} = c_Q(Q) = c_Q(2 \bar{H})$ and the total class of the normal bundle is $c(N_{Q, \mathbb{P}^{n+1}}) = 1 + 2 \bar{H}$. On the other
hand, it is well known that $c(T_{P^{d+1}}) = (1 + H)^{d+1}$. Now, from the splitting principle the claim follows.

Let us apply this lemma to the previous situation. We obtain

$$c(N_{A}, q) = (1 + h)^{2d+2}(1 + 2h)^{-1} = \sum_{k=0}^{2d+2} \binom{2d+2}{k} h^{k} \sum_{l=0}^{n} (-2h)^{l}$$

where $h = j^* H$. In particular, the top class is

$$c_d = F_d h^d, \quad \text{with} \quad F_d = \sum_{k=0}^{d} \binom{2d+2}{k} (-2)^{(d-k)}.$$ 

Substituting this into the self-intersection formula, we have:

$$A.A = F_d j^* (j^* H^d) = F_d H^d, \quad j^* A = F_d (a \alpha + b \beta), \quad H^d = F_d (a + b).$$ 

Combining this expression with (1) we obtain the following relation

(2) $$a^2 + b^2 = F_d (a + b)$$

or equivalently,

$$\left( a - \frac{F_d}{2} \right)^2 + \left( b - \frac{F_d}{2} \right)^2 = \frac{F_d^2}{2}.$$ 

We are interested in bounding the degree of $A$, when $(a, b)$ satisfy this equation. Note that this is a circle of center $\left( \frac{F_d}{2}, \frac{F_d}{2} \right)$ and radius $\frac{F_d}{\sqrt{2}}$.

Since $\deg(A) = a + b$, it is clear that the maximal degree is reached when $(a, b) = (F_d, F_d)$, that is,

$$\deg(A) \leq 2F_d.$$ 

On the other hand, the abelian variety is embedded in $Q \subset P^{2d+1}$. When $d \geq 2$, by Van de Ven's Theorem, it spans $P^{2d+1}$. Furthermore, by the Riemann-Roch theorem for abelian varieties, we know that $h^0(\mathcal{O}_A(h)) = \deg(A)$. Thus, we have the following inequality:

$$\deg(A) \geq 2(d+1)!$$

Comparing the two bounds we see that a sufficient condition for the non-
existence of the embedding $j$ is $F_d < (d + 1)!$. Now,

$$F_d = \sum_{k=0}^{d} \binom{2d + 2}{k} (-2)^{d-k} \leq \sum_{k=0}^{d} \binom{2d + 2}{k} (2)^{d} \leq 2^{d+1} = 2^{d+1}.$$ 

We see that $(d + 1)! > 2^{d+1}$ for $d = 17$. A simple inductive argument shows that this holds if $d \geq 17$.

If $d \leq 17$, using the exact value of $F_d$, we see that $(d + 1)! > F_d$ for any $d > 3$.

We conclude that the unique possibilities are $d = 2$ or $d = 3$.

First, suppose that $A$ is an abelian surface contained in a quadric $F_2 = 7$ and we can check that the unique positive integer solution of the equation (2) is $a = b = 7$. Thus $A$ must be an abelian surface of degree 14 given by the polarization $(1, 7)$. Note that $A \subset Q \subset P^5$ is not linearly normal, that is, it is the projection of a linearly normal abelian surface $A \subset P^6$. The quadric $Q$ can be lifted to a quadric containing the surface $A$.

Lazarsfeld proved in [7] that a very ample divisor of type $(1, d)$ with $d \geq 13$ or $d = 6, 8, 9$ is projectively normal. From this the following sequence is exact:

$$0 \rightarrow H^0(I_A, \mathcal{O}_P(2)) \rightarrow H^0(\mathcal{O}_P(2)) \rightarrow H^0(\mathcal{O}_A(2)) \rightarrow 0.$$ 

Since $h^0(\mathcal{O}_P(2)) = h^0(\mathcal{O}_A(2)) = 28$, there are no quadrics containing the abelian surface $A'$ and we obtain a contradiction.

Finally, suppose that $d = 3$. Now, $F_3 = 24 = (3 + 1)!$, so the degree of the abelian variety is exactly $2F_3 = 48$. The line bundle $\mathcal{O}_A(h)$ corresponds to a divisor of type $(1, 1, 8)$ or $(1, 2, 4)$. But J.N. Iyer prove in [5] that a line bundle of type $(1, \ldots, 1, 2d + 1)$ is never very ample. Moreover, in [6] she studies the map defined by a line bundle of type $(1, 2, 4)$ in a generic abelian threefold. She obtains that it is birational but not an isomorphism onto its image. Note that the very ampleness is an open condition for polarized abelian varieties (see [2]). It follows that a linear system of type $(1, 2, 4)$ cannot be very ample on any abelian threefold and this completes the proof.

**Remark 2.2.** In [1] C. Ciliberto and V. Di Gennaro obtain more general results about subvarieties of low codimension. In particular they give a bound for the degree of a $d$-dimensional abelian variety embedded on a smooth hypersurface of dimension $2d$. 


Remark 2.3. The sequence $F_d = \sum_{k=0}^{d} \binom{2d+k}{k} (-2)^{d-k}$ is related to the Fine numbers. For a reference see [3].

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References


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